ADDITIVE GROTHENDIECK PRETOPOLOGIES AND PRESENTATIONS OF TENSOR CATEGORIES

KEVIN COULEMBIER

ABSTRACT. We study how tensor categories can be presented in terms of rigid monoidal categories and Grothendieck topologies and show that such presentations lead to strong universal properties. As the main tool in this study, we define a notion on preadditive categories which plays a role similar to the notion of a Grothendieck pretopology on an unenriched category. Each such additive pretopology defines an additive Grothendieck topology and suffices to define the sheaf category. This new notion also allows us to study the noetherian and subcanonical nature of topologies, to describe easily the meet of a family of topologies and to identify useful universal properties of the sheaf category.

INTRODUCTION

The notion of a Grothendieck topology on a small category **c** allows one to define a category of sheaves and the development of sheaf cohomology, most notably étale cohomology on a scheme, see [AGV, Ta]. According to the original terminology, a Grothendieck topology is a collection of 'sieves' satisfying suitable axioms. A Grothendieck *pre*topology is a collection of 'coverings' $\{U_i \rightarrow U \mid i\}$, generalising the properties of open coverings of topological spaces. Each pretopology induces a topology, but many pretopologies induce the same topology.

It has become customary to refer to pretopologies simply as topologies, see [Ta]. From some points of view it can indeed be preferable to work with pretopologies. A presheaf $F: \mathbf{c}^{\text{op}} \to \mathsf{Set}$ is a sheaf for the induced topology if and only if, for every covering $\{U_i \to U\}$ in the pretopology, the diagram

$$F(U) \to \prod_{i} F(U_i) \Rightarrow \prod_{j,l} F(U_j \times_U U_l)$$
(1)

is exact. This recovers the familiar definition of a sheaf on a topological space.

The notion of Grothendieck topologies extends canonically to enriched categories, see [BQ]. Furthermore, by [BQ] and the Gabriel-Popescu theorem [PG], a category is Grothendieck abelian if and only if it can be realised as the category of additive sheaves on a preadditive site. Already for preadditive categories, the notion of a Grothendieck pretopology does not extend naively. However, there are some examples of additive Grothendieck topologies where one can (and usually does) define sheaves with respect to exactness of certain sequences, similar to the classical sheaf condition (1). Firstly, let **a** be a small abelian category. The ind-completion Ind**a** can be realised as the category of left exact functors $\mathbf{a}^{\text{op}} \rightarrow \mathbf{Ab}$. Secondly, we can start from a non-enriched presite **c** and consider the preadditive category $\mathbb{Z}\mathbf{c}$ freely generated by **c**. By taking the difference of the two canonical maps we obtain, for every covering $\{U_i \rightarrow U\}$, a sequence

$$\amalg_{i,l} U_i \times_U U_l \to \amalg_i U_i \to U \to 0,$$

where the coproducts are 'formal' unless \mathbf{c} admits coproducts of sufficiently large cardinality. The Grothendieck category of abelian sheaves on \mathbf{c} can be realised as the category of additive functors $\mathbb{Z}\mathbf{c}^{\mathrm{op}} \rightarrow \mathsf{Ab}$ which send the above sequences to exact ones in Ab .

²⁰²⁰ Mathematics Subject Classification. 18E10, 18E35, 18F10, 18D15.

Key words and phrases. Additive Grothendieck topology, Grothendieck category, noetherian and subcanonical topologies, tensor category.

KEVIN COULEMBIER

We therefore define an additive pretopology on \mathbf{a} to be a class of formal sequences in \mathbf{a}

$$\amalg_{\gamma} Z_{\gamma} \to \amalg_{\beta} Y_{\beta} \to X_{\gamma}$$

which satisfies two simple conditions. Our definition is much broader than what is needed to incorporate our two guiding examples above. The main advantage is that the union of two pretopologies will again be a pretopology. This feature, which is not satisfied for unenriched pretopologies, is useful for several applications.

We prove that each additive pretopology yields an additive topology and that every additive topology comes from some additive pretopology. Furthermore, sheaves with respect to the topology correspond precisely to additive functors $F : \mathbf{a}^{\text{op}} \to \mathsf{Ab}$ for which

$$0 \to F(X) \to \prod_{\beta} F(Y_{\beta}) \to \prod_{\gamma} F(Z_{\gamma})$$

is exact for all sequences in the pretopology.

For the specific case of additive topologies which are both noetherian and subcanonical (as defined below) on a category **a** which moreover is additive, our theory essentially recovers the theory of 'ind-classes' as developed by Schäppi in [Sc1, Sc2]. Our theory is directly inspired by the latter, and this is in particular the case for many of the techniques in Section 2.3.

Part of the motivation for the above constructions comes from the theory of tensor categories, in the sense of [De, EGNO], more specifically the theory of abelian envelopes, see [EHS, Co2] and references therein. We investigate in which ways one can present a tensor category (or more precisely its ind-completion) via a generating rigid monoidal category and a monoidal Grothendieck topology. Using the above techniques we can reduce this to a manageable set of topologies and show that they come with strong universal properties, which will be crucial for applications in work in preparation. The main results here are Corollary 4.4.4, which significantly generalises [EHS, Theorem 9.2.2], and Theorem 4.4.1.

The paper is organised as follows. In Section 1 we recall the required background on Grothendieck topologies and categories. In Section 2 we introduce additive pretopologies, establish the connection with additive topologies, prove the characterisation of sheaves and demonstrate that the notion of a pretopology provides a universal property for the sheaf category. In Section 3 we study subcanonical and noetherian topologies, as inspired by the corresponding unenriched notions in [Ta, §1.3 and §3.10]. A topology is subcanonical if the representable presheaves are sheaves and noetherian if the sheaffication of the representable presheaves are compact. These notions are difficult to characterise directly from the topology, but are easily described on the level of pretopologies. We also introduce the notion of monoidal Grothendieck topologies. We end the section with a discussion of the 'tensor product' of two Grothendieck categories from [LRS]. In Section 4 we investigate the presentations of tensor categories.

1. Preliminaries

We set $\mathbb{N} = \{0, 1, 2, ...\}$. All categories we will consider are locally small.

1.1. Grothendieck categories.

1.1.1. An object G in a category C is a **generator** if $C(G, -) : C \to Set$ is faithful. More generally, we call an essentially small full subcategory $c \in C$ a generator if $\prod_{A \in c} C(A, -)$ is faithful.

A **Grothendieck category** \mathbf{C} is an AB5 abelian category which admits a generator. In other words, \mathbf{C} is abelian with a generator, \mathbf{C} admits set-indexed coproducts, and filtered colimits of short exact sequences in \mathbf{C} are exact. A Grothendieck category has enough injective objects and by Freyd's special adjoint functor theorem, a cocontinuous functor out of a Grothendieck category has a left adjoint.

For every essentially small preadditive category \mathbf{a} , the presheaf category PSha of additive functors $\mathbf{a}^{\mathrm{op}} \to \mathsf{Ab}$ is a Grothendieck category, with generator \mathbf{a} (identified with a full subcategory via the Yoneda embedding $Y : \mathbf{a} \to \mathsf{PSha}$).

1.1.2. Localisations. Let **B** be a Grothendieck category. In the literature the notions of 'localisation' and 'Giraud subcategory' of **B** appear interchangeably. It will be convenient to use them for slightly distinct structures. A fully faithful additive functor $\mathbf{I} : \mathbf{C} \hookrightarrow \mathbf{B}$, from a preadditive category **C** is **reflective** if it has a left adjoint $\mathbf{S} : \mathbf{B} \to \mathbf{C}$, which we refer to as the sheafification. If **I** is reflective, it follows that **C** is cocomplete.

A reflective functor I is a **localisation** if S is (left) exact. If I is a localisation, it follows that C is a Grothendieck category itself. By a **Giraud subcategory** of B we mean a full replete subcategory for which the inclusion is a localisation. In other words, Giraud subcategories correspond to equivalence classes of localisations.

Besides the notation I and S for the inclusion and sheafification of a reflective subcategory C of PSha, we will fix the notation Z for the functor

$$\mathsf{Z} = \mathsf{S} \circ \mathsf{Y} : \mathbf{a} \to \mathbf{C}.$$

By the Gabriel-Popescu theorem [PG], a Grothendieck category \mathbf{C} with generator $G \in \mathbf{C}$ is a localisation of Mod_R , for the ring $R \coloneqq \mathbf{C}(G, G)$. It is often convenient to consider a slight generalisation, which recovers the original theorem for a one object category \mathbf{c} .

Theorem 1.1.3 (Gabriel - Popescu). For a Grothendieck category \mathbf{C} with generator $\mathbf{c} \in \mathbf{C}$, the functor $\mathbf{C} \to \mathsf{PShc} : M \mapsto \mathbf{C}(-, M)|_{\mathbf{c}}$ is a localisation of PShc .

1.1.4. Creators. We refer to any additive functor $u : \mathbf{a} \to \mathbf{C}$ from an essentially small preadditive category \mathbf{a} , for which the right adjoint $\mathbf{C} \to \mathsf{PSha}$: $M \mapsto \mathbf{C}(u-, M)$ is a localisation (so in particular fully faithful) as a **creator**. In [Lo] these functors are intrinsically characterised and an obvious necessary condition is that the essential image of u is a generator in \mathbf{C} . Theorem 1.1.3 can then be rephrased as saying that the embedding of a generator is a creator.

1.2. Notation and conventions.

1.2.1. Since many Grothendieck categories of interest are k-linear for some commutative ring k we will henceforth work over an unspecified commutative ring k. We will also denote by **a** an essentially small k-linear category.

For two k-linear categories \mathbf{B}, \mathbf{C} , we denote by $[\mathbf{B}, \mathbf{C}]$ the k-linear category of k-linear functors $\mathbf{B} \to \mathbf{C}$. In particular, we set

$$\mathsf{PSha} \coloneqq [\mathbf{a}^{\mathrm{op}}, \mathsf{Mod}_k].$$

We will typically leave out reference to k, so 'functor', 'sieve' and '(pre)topology' will refer to the k-linear versions (with the obvious exceptions of Sections 1.5 and 2.4). We will use the symbol \oplus for coproducts and reserve the \amalg for 'formal coproducts'.

1.2.2. Recall that a cardinal κ is **regular** if the class of sets of cardinality strictly lower than κ is closed under taking unions of strictly fewer than κ sets. We only assume that a regular cardinal is bigger than 1. In other words, we consider the regular cardinal 2 and infinite regular cardinals.

1.3. Compact objects. Let κ be an infinite regular cardinal. Our main case of interest will be $\kappa = \aleph_0$, in which case we omit κ from terminology. Let **C** be a Grothendieck category. An object $X \in \mathbf{C}$ is κ -compact (resp. κ -generated) if $\mathbf{C}(X, -) : \mathbf{C} \to \mathsf{Ab}$ commutes with all κ -filtered colimits (resp. colimits of κ -filtered diagrams consisting of monomorphisms).

Lemma 1.3.1. (i) Let $A \in \mathbb{C}$ be κ -generated. For every epimorphism $\bigoplus_{\beta \in B} M_{\beta} \twoheadrightarrow A$ in \mathbb{C} , there exists $B_0 \subset B$ with $|B_0| < \kappa$ such that $\bigoplus_{\beta \in B_0} M_{\beta} \twoheadrightarrow A$ is still an epimorphism.

(ii) Let $A \in \mathbf{C}$ be κ -compact. For every exact sequence

$$\bigoplus_{\gamma \in C} N_{\gamma} \to \bigoplus_{\beta \in B} M_{\beta} \to A \to 0$$

in **C**, there exist subsets $B_0 \subset B$ and $C_0 \subset C$ with $|B_0| < \kappa > |C_0|$ such that restricting the summations to these subsets still yields an exact sequence.

(iii) The presheaf $F \in \mathsf{PSha}$ is κ -compact if and only if there exists an exact sequence

$$\bigoplus_{\gamma \in C} \mathbb{Y}(Z_{\gamma}) \to \bigoplus_{\beta \in B} \mathbb{Y}(Y_{\beta}) \to F \to 0$$

with $|B| < \kappa > |C|$.

(iv) The presheaf $F \in \mathsf{PSha}$ is κ -generated if and only if there exists an epimorphism $\bigoplus_{\beta \in B} \mathtt{Y}(Y_{\beta}) \twoheadrightarrow F$ with $|B| < \kappa$.

Proof. For each $B' \subset B$ with $|B'| < \kappa$ denote by $A_{B'}$ the image of $\bigoplus_{\beta \in B'} M_{\beta} \to A$. Then A is the colimit of these subobjects, so part (i) follows by definition. By part (i) we can find appropriate $B_0 \subset B$ in part (ii). Subsequently, we can realise A as a colimit of the cokernels of $\bigoplus_{\gamma \in C'} N_{\gamma} \to \bigoplus_{\beta \in B_0} M_{\beta}$ for all $|C'| < \kappa$, to prove we can find C_0 .

One direction of parts (iii) and (iv) follows from parts (i) and (ii). The other directions follow by using the fact that in Ab, κ -filtered colimits commute with κ -small limits.

1.4. Linear Grothendieck topologies. We briefly review the theory of enriched topologies from [BQ], for the case of k-linear enrichment.

1.4.1. For $A \in \mathbf{a}$, a (k-linear) **sieve on** A is a k-linear subfunctor of $\mathbf{a}(-, A) \in \mathsf{PSha}$. For a sieve R on A and a morphism $f : B \to A$ in \mathbf{a} , we denote by $f^{-1}R$ the sieve on B which is the pullback of $R \to \mathbf{a}(-, A) \leftarrow \mathbf{a}(-, B)$ in PSha. Concretely, we have

 $g \in f^{-1}R(C) \Leftrightarrow f \circ g \in R(C),$ for all $C \in A$ and $g \in \mathbf{a}(C, B)$.

A covering system \mathcal{T} on **a** is an assignment to each $A \in \mathbf{a}$ of a set $\mathcal{T}(A)$ of sieves on A.

Definition 1.4.2. A k-linear Grothendieck **topology** is a covering system \mathcal{T} on **a** such that for every $A \in \mathbf{a}$:

- (T1) We have $\mathbf{a}(-, A) \in \mathcal{T}(A)$;
- (T2) For $R \in \mathcal{T}(A)$ and a morphism $f: B \to A$ in **a**, we have $f^{-1}R \in \mathcal{T}(B)$;
- (T3) For a sieve S on A and $R \in \mathcal{T}(A)$ such that for every $B \in \mathbf{a}$ and $f \in R(B) \subset \mathbf{a}(B, A)$ we have $f^{-1}S \in \mathcal{T}(B)$, it follows that $S \in \mathcal{T}(A)$.

A direct consequence of (T1) and (T3) is the following property:

(T4) For a sieve S on A and $R \in \mathcal{T}(A)$ such that $R \subset S$, it follows that $S \in \mathcal{T}(A)$. Similarly, a consequence of (T2) and (T3) is:

(T5) If $R_1, R_2 \in \mathcal{T}(A)$, then $R_1 \cap R_2 \in \mathcal{T}(A)$.

Grothendieck topologies on **a** are the same when regarding **a** as a k-linear category or as a preadditive (\mathbb{Z} -linear) category. Moreover PSha is equivalent to the category of additive functors $\mathbf{a}^{\text{op}} \rightarrow A\mathbf{b}$. So the k-linearity does not play any role here, it is just useful to take it along for specific applications.

1.4.3. The class of covering systems, so also the set of Grothendieck topologies, is ordered by inclusion. We say that \mathcal{T}_1 is a refinement of \mathcal{T} if $\mathcal{T} \subset \mathcal{T}_1$, which means that $\mathcal{T}(A) \subset \mathcal{T}_1(A)$ for all $A \in \mathbf{a}$. It is clear that for a family of Grothendieck topologies $\{\mathcal{T}_i\}$ the covering system $\cap_i \mathcal{T}_i$ is again a Grothendieck topology. The same is not true for $\cup_i \mathcal{T}_i$, even when the family is finite. However, by the previous property, we have a well-defined notion of the minimal (coarsest) Grothendieck topology containing all \mathcal{T}_i , which is denoted by $\vee_i \mathcal{T}_i$. **Definition 1.4.4.** For a k-linear Grothendieck topology \mathcal{T} on **a**, a presheaf $F \in \mathsf{PSha}$ is a \mathcal{T} -sheaf if for every $A \in \mathbf{a}$ and $R \in \mathcal{T}(A)$, the morphism induced from $R \hookrightarrow \mathbf{a}(-, A)$

$$F(A) \simeq \operatorname{Nat}(\mathbf{a}(-, A), F) \to \operatorname{Nat}(R, F)$$

is an isomorphism. The full subcategory of PSha of \mathcal{T} -sheaves is denoted by $\mathsf{Sh}(\mathbf{a},\mathcal{T})$.

When $F(A) \rightarrow \operatorname{Nat}(R, F)$ is a monomorphism for every $A \in \mathbf{a}$ and $R \in \mathcal{T}(A)$, we say that the presheaf F is \mathcal{T} -separated.

The following theorem can be extracted from [BQ, Theorem 1.5] and its proof.

Theorem 1.4.5 (Borceux - Quinteiro).

- (i) For each k-linear Grothendieck topology \mathcal{T} on \mathbf{a} , the subcategory $\mathsf{Sh}(\mathbf{a}, \mathcal{T})$ is a localisation of PSha .
- (ii) For each localisation $I : C \hookrightarrow \mathsf{PSha}$, the covering system \mathcal{T} on \mathbf{a} given by all sieves $R \subset \mathbf{a}(-, X)$ which satisfy one of the equivalent conditions
 - (a) $\operatorname{Nat}(\mathbf{a}(-,X), \mathbf{I}M) \to \operatorname{Nat}(R, \mathbf{I}M)$ is an isomorphism for every $M \in \mathbf{C}$;
 - (b) S maps $R \hookrightarrow Y(X)$ to an isomorphism;
 - is a k-linear Grothendieck topology.

The above procedures give mutually inverse bijections between the set of k-linear Grothendieck topologies on \mathbf{a} and the set of Giraud subcategories of PSha.

1.4.6. Sheafification. Fix a Grothendieck topology \mathcal{T} on **a**. For $X \in \mathbf{a}$, the partial order \leq on $\mathcal{T}(X)$ given by $R \leq R'$ if $R' \subset R$ is directed by (T5). By [BQ, Theorem 4.1], we have an endofunctor Σ of PSha such that

$$\Sigma F(X) = \lim_{R \in \mathcal{T}(X)} \operatorname{Nat}(R, F), \quad \text{for } F \in \mathsf{PSha} \text{ and } X \in \mathbf{a}$$
(2)

and by [BQ, Theorem 4.4] we have $\mathbf{I} \circ \mathbf{S} = \Sigma \circ \Sigma$. There is an obvious natural transformation $\sigma : \mathrm{id} \to \Sigma$, and the unit $\eta : \mathrm{id} \to \mathbf{I} \circ \mathbf{S}$ of the adjunction $\mathbf{S} \dashv \mathbf{I}$ corresponds to $(\Sigma \sigma) \circ \sigma : \mathrm{id} \to \Sigma \circ \Sigma$. Other elements of [BQ, Theorems 4.1 and 4.4] we need are:

- (i) Σ : PSha \rightarrow PSha is left exact;
- (ii) $F \in \mathsf{PSha}$ is a sheaf if and only if $\sigma_F : F \to \Sigma F$ is an isomorphism;
- (iii) $F \in \mathsf{PSha}$ is separated if and only if $\sigma_F : F \to \Sigma F$ is a monomorphism;
- (iv) ΣF is separated for every presheaf F;
- (v) ΣF is a sheaf, when F is separated.

1.5. Non-enriched pretopologies. Definition 1.4.2 is a direct analogue of the classical definition of a non-enriched Grothendieck topology in [AGV, Définition II.1.1]. We recall the notion of a pretopology from [AGV, Définition II.1.3].

Definition 1.5.1. A Grothendieck **pretopology** on a category **c** is an assignment to each $U \in \mathbf{c}$ of a collection of sets of morphisms $\{U_i \rightarrow U\}$, called coverings of U, such that:

- (pt0) If $\{U_i \to U\}$ is a covering and $f: V \to U$ any morphism, then all $U_i \times_U V$ exist.
- (pt1) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.
- (pt2) If $\{U_i \to U\}$ is a covering and $f: V \to U$ any morphism in **c**, then $\{U_i \times_U V \to V\}$ is a covering.
- (pt3) If $\{U_i \to U\}$ and $\{V_{ij} \to U_i\}$ are coverings, then $\{V_{ij} \to U_i \to V\}$ is a covering.

1.5.2. The basic fact that, for every $A \in \mathbf{c}$, the diagram

$$\bigsqcup_{kl} \mathbf{c}(A, U_k \times_U U_l) \ \Rightarrow \ \bigsqcup_i \mathbf{c}(A, U_i) \to \mathbf{c}(A, U) \tag{3}$$

is exact plays an important role in the connection between (nonenriched) pretopologies and topologies, see for instance the proof of [AGV, Corollaire II.2.4].

KEVIN COULEMBIER

On the other hand, for morphisms of abelian groups $\{B_i \rightarrow B \mid i \in I\}$, the sequence

$$\bigoplus_{kl} B_k \times_B B_l \to \bigoplus_i B_i \to B$$

with the first morphism the difference of the two projections, is not necessarily exact if |I| > 2. Hence, in a preadditive category we cannot expect the additive analogue of (3) to be exact. This explains our need to deviate from Definition 1.5.1 for additive pretopologies.

2. Pretopologies

2.1. Formal sequences.

2.1.1. We will consider morphisms between formal coproducts of objects in **a**. These are actual morphisms in PSha or in the formal completion of **a** under small coproducts. In other words, (formal) morphisms $f : \amalg_{\alpha} A_{\alpha} \to \amalg_{\beta} B_{\beta}$, for objects $A_{\alpha}, B_{\beta} \in \mathbf{a}$, are elements

$$f = (f_{\alpha\beta}) \in \prod_{\alpha} \bigoplus_{\beta} \mathbf{a}(A_{\alpha}, B_{\beta}).$$

The composition $g \circ f$ for $g: \amalg_{\beta} B_{\beta} \to \amalg_{\gamma} C_{\gamma}$ is defined by $(g \circ f)_{\alpha\gamma} = \sum_{\beta} g_{\beta\gamma} \circ f_{\alpha\beta}$, where by definition the sum is finite and, for a fixed α , we have $(g \circ f)_{\alpha\gamma} = 0$ for all but finitely many γ .

For a morphism $v = (v_{\alpha}) : \amalg_{\alpha} V_{\alpha} \to X$, we denote the sieve on X generated by all v_{α} by $R_v \subset \mathbf{a}(-, X)$. Clearly every sieve can be written in this way.

2.1.2. We are mainly interested in pairs (p,q) of formal morphisms

$$I_{\gamma} Z_{\gamma} \xrightarrow{p} \amalg_{\beta} Y_{\beta} \xrightarrow{q} X, \tag{4}$$

with $q \circ p = 0$. We will refer to them as (formal) sequences. The sequence (4) is **right exact** if the induced sequence in Mod_k

$$\prod_{\gamma} \mathbf{a}(Z_{\gamma}, A) \leftarrow \prod_{\beta} \mathbf{a}(Y_{\beta}, A) \leftarrow \mathbf{a}(X, A) \leftarrow 0$$

is exact for every $A \in \mathbf{a}$. Similarly we say that q is an epimorphism if $\mathbf{a}(X, A) \to \prod_{\beta} \mathbf{a}(Y_{\beta}, A)$ is injective for all A. These notions of exactness do **not** correspond to exactness in PSha. If the sequence is not formal ('2-bounded' in the terminology below), we recover the ordinary notion of cokernels and epimorphisms in \mathbf{a} (which differ from the ones in PSha).

For a regular cardinal κ , we call formal morphisms and sequences where the labelling sets in the coproducts are strictly bounded by κ in cardinality, κ -**bounded**. Similarly, a class of sequences is called κ -bounded if every sequence it contains is κ -bounded.

2.1.3. For a class S of sequences (4), we denote by Co(S) the class of morphisms q which appear on the right of the sequences. For $X \in \mathbf{a}$, we denote by $Co_X(S) \subset Co(S)$ the subclass of morphisms with target X.

Set $\mathcal{C}o^1(\mathcal{S}) = \mathcal{C}o(\mathcal{S})$ and denote by $\mathcal{C}o^0(\mathcal{S})$ the class of all identity morphisms in **a**. For i > 1 we define $\mathcal{C}o^i(\mathcal{S})$ iteratively as follows. If $v : \amalg_{\alpha} V_{\alpha} \to X$ is in $\mathcal{C}o^{i-1}(\mathcal{S})$ and for each α we have $w(\alpha) : \amalg_{\delta} W_{\delta}(\alpha) \to V_{\alpha}$ in $\mathcal{C}o(\mathcal{S})$, then the collection of morphism $v_{\alpha} \circ w(\alpha)_{\delta}$ form $\amalg_{\alpha} \amalg_{\delta} W_{\delta} \to X$ in $\mathcal{C}o^i(\mathcal{S})$. We also set

$$\widetilde{\mathcal{C}o}(\mathcal{S}) = \bigcup_{i \in \mathbb{N}} \mathcal{C}o^i(\mathcal{S}).$$

We also let $Sh_{\mathcal{S}}a$ be the full subcategory of PSha of all F for which

$$0 \to F(X) \to \prod_{\beta} F(Y_{\beta}) \to \prod_{\gamma} F(Z_{\gamma})$$

is exact in Mod_k for every $\amalg_{\gamma} Z_{\gamma} \to \amalg_{\beta} Y_{\beta} \to X$ in \mathcal{S} .

Lemma 2.1.4. Let S be a κ -bounded class of sequences (4), for an infinite regular cardinal κ . Consider the inclusion $I : Sh_{S}a \hookrightarrow PSha$.

- (i) For every functor $J : \mathbf{j} \to \mathsf{Sh}_{\mathcal{S}}\mathbf{a}$ from a κ -filtered category \mathbf{j} , the canonical morphism $\operatorname{colim}(\mathbf{I} \circ J) \to \mathbf{I}(\operatorname{colim} J)$ is an isomorphism.
- (ii) The inclusion I is reflective.
- (iii) The image of $Z : a \to Sh_{\mathcal{S}}a$ consists of κ -compact objects.

Proof. Since κ -small limits commute with κ -filtered colimits in Mod_k , it follows that the presheaf $\mathsf{colim}(\mathfrak{I} \circ J)$ is actually contained in $\mathsf{Sh}_{\mathcal{S}}\mathbf{a}$. Part (i) is then a standard consequence. Part (ii) follows from the main result in [AR] by part (i). For part (iii), we observe that for $X \in \mathbf{a}$ and a functor J as in (i), we have

 $\mathsf{Sh}_{\mathcal{S}}\mathbf{a}(\mathsf{Z}(X), \operatorname{colim} J) \simeq \mathsf{PSha}(\mathsf{Y}(X), \mathtt{I}(\operatorname{colim} J)) \simeq \operatorname{colim}(\mathtt{I} \circ J)(X) \simeq \operatorname{colim}\mathsf{Sh}_{\mathcal{S}}\mathbf{a}(\mathsf{Z}(X), J),$

which shows that Z(X) is κ -compact.

2.2. Pretopologies versus topologies.

Definition 2.2.1. A *k*-linear Grothendieck pretopology on **a** is a class S of sequences of the form (4) such that (PTa) and (PTb) are satisfied:

(PTa) For each $q: \amalg_{\beta} Y_{\beta} \to X$ in $\mathcal{C}o(\mathcal{S})$ and morphism $f: A \to X$, there exists $q' \in \widetilde{\mathcal{C}o}_A(\mathcal{S})$ which admits a (formal) commutative diagram

$$\begin{array}{c} \amalg_{\beta}Y_{\beta} \xrightarrow{q} X \\ f' & f \\ \downarrow \\ \amalg_{\delta}C_{\delta} - - \xrightarrow{q'} - - \succ A. \end{array}$$

(PTb) For every sequence (4) in S and $f : A \to \coprod_{\beta} Y_{\beta}$ with $q \circ f = 0$, there exists $q' \in \widetilde{\mathcal{Co}}_A(S)$ which admits a (formal) commutative diagram

Remark 2.2.2. The following are potential properties of a class S of sequences (4): (PTb') For every sequence (4) in S and $A \in \mathbf{a}$, the following sequence in Mod_k is exact:

$$\bigoplus_{\gamma} \mathbf{a}(A, Z_{\gamma}) \to \bigoplus_{\beta} \mathbf{a}(A, Y_{\beta}) \to \mathbf{a}(A, X).$$

(PTc) For every $r: \amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{\mathcal{Co}}(\mathcal{S})$ and $f: A \to X$ in \mathbf{a} , there exists $r' \in \widetilde{\mathcal{Co}}_A(\mathcal{S})$ which admits a (formal) commutative diagram

$$\begin{array}{c} \amalg_{\alpha}V_{\alpha} \xrightarrow{r} X \\ \stackrel{h}{\downarrow} & f \\ \downarrow \\ \amalg_{\delta}B_{\delta} - - \xrightarrow{r'} - - \gg A. \end{array}$$

It is clear that (PTb') implies (PTb). Furthermore, as proved below, (PTa) implies (PTc), so (PTc) is satisfied for every pretopology.

Lemma 2.2.3. Consider a collection $\{\amalg_{\beta}Y_{\beta}^{i} \xrightarrow{q^{i}} X^{i} | i \in I\}$ of elements in $\mathcal{C}o(\mathcal{S})$.

(i) For a morphism $f = (f_i) : A \to \coprod_i X^i$, there exists $q' \in \widetilde{Co}_A(S)$ which admits a commutative diagram as in (PTa) with each pair (q^i, f_i) .

(ii) Condition (PTc) is satisfied for S.

Proof. By the definition of formal morphisms $f : A \to \sqcup_i X^i$, it is sufficient to consider finite collections I in (i). We then prove (i) by induction on |I|. The base case |I| = 1 is precisely (PTa). Now assume that we have proved the claim for $\{q^i : \amalg_\beta Y^i_\beta \to X^i | 1 \le i < n\}$ and consider $q^n : \amalg_\beta Y^n_\beta \to X^n$ in $\mathcal{Co}_X(\mathcal{S})$. By assumption, we have a commutative diagram

for some $q'' \in \widetilde{\mathcal{Co}}_A(\mathcal{S})$. We can now apply (PTa) to q^n and each of the $f^n \circ q''_{\delta} : C_{\delta} \to X^n$ to get a suitable element of $\widetilde{\mathcal{Co}}_{C_{\delta}}(\mathcal{S})$. Part (i) allows us to prove (PTc) in part (ii) iteratively. \Box

The class of pretopolgies is canonically ordered with respect to inclusion. It follows immediately from the definition that for a family of pretopologies $\{S_i\}$, the class of sequences $\cup_i S_i$ is again a pretopology. However, for pretopologies S_1 and S_2 , the class of sequences $S_1 \cap S_2$ need not be a pretopology. This behaviour is dual to that of topologies, see 1.4.3. We will exploit this in Theorems 2.2.5(v) and 3.6.2 below.

2.2.4. For a pretopology S we consider the covering system top(S) of all sieves $R \subset \mathbf{a}(-, X)$ which contain R_r for some $r \in \widetilde{\mathcal{Co}}_X(S)$.

For a topology \mathcal{T} on **a**, we denote by $\operatorname{pre}'(\mathcal{T})$ the class of all formal sequences (4) for which the induced sequence

$$\bigoplus_{\gamma} \mathbb{Y}(Z_{\gamma}) \to \bigoplus_{\beta} \mathbb{Y}(Y_{\beta}) \to \mathbb{Y}(X)$$
(5)

is exact in PSha such that the image of the right morphism is an element in $\mathcal{T}(X)$. The class pre' has the advantage of being defined directly from $(\mathbf{a}, \mathcal{T})$. However, we will also need the class pre (\mathcal{T}) of all formal sequences (4) for which the induced sequence

$$\bigoplus_{\gamma} \mathsf{Z}(Z_{\gamma}) \to \bigoplus_{\beta} \mathsf{Z}(Y_{\beta}) \to \mathsf{Z}(X) \to 0$$

is exact in $Sh(\mathbf{a}, \mathcal{T})$.

More generally, consider an exact cocontinuous functor Θ : PSha \rightarrow C to a cocomplate abelian category C. We denote by $\mathcal{S}(\Theta)$ the class of formal sequences (4), for which Θ sends (5) to an exact sequence in C. We thus have pre(\mathcal{T}) = $\mathcal{S}(S)$.

Theorem 2.2.5. Let \mathcal{T} be a topology and \mathcal{S} a pretopology on \mathbf{a} .

- (i) The classes of sequences $pre(\mathcal{T})$ and $pre'(\mathcal{T})$ are pretopologies.
- (ii) The covering system $top(\mathcal{S})$ is a topology.
- (*iii*) We have $\operatorname{top}(\operatorname{pre}'(\mathcal{T})) = \mathcal{T} = \operatorname{top}(\operatorname{pre}(\mathcal{T})).$
- (iv) The operations pre' and top are order (inclusion) preserving.
- (v) For a family of pretopologies $\{S_i | i \in I\}$, set $\mathcal{T}_i = \operatorname{top} S_i$. Then $\lor_{i \in I} \mathcal{T}_i = \operatorname{top} (\cup_{i \in I} S_i)$, so in particular

$$\vee_{i\in I}\mathcal{T}_i = \operatorname{top}(\cup_{i\in I}\operatorname{pre}(\mathcal{T}_i)).$$

We start the proof of the theorem with the following lemma.

Lemma 2.2.6. For a topology \mathcal{T} and a pretopology \mathcal{S} , we have $\operatorname{top}(\mathcal{S}) \subset \mathcal{T}$ if and only if $R_q \in \mathcal{T}(X)$ for every $q \in \mathcal{C}o_X(\mathcal{S})$ and $X \in \mathbf{a}$.

Proof. One direction is obvious. To prove the other direction, we assume that $R_q \in \mathcal{T}(X)$ for every $q \in \mathcal{C}o(\mathcal{S})$. By (T4) it suffices to show that $R_r \in \mathcal{T}(X)$ for every $r \in \widetilde{\mathcal{C}o_X}(\mathcal{S})$. We prove this by induction on i in $\widetilde{\mathcal{C}o}(\mathcal{S}) = \cup_i \mathcal{C}o^i(\mathcal{S})$, where i = 0 is fine by (T1) and i = 1 is fine by assumption. Consider $v : \amalg_{\alpha} V_{\alpha} \to X$ in $\mathcal{C}o^i(\mathcal{S})$, and for each α we take $q(\alpha)$ in $\mathcal{C}o_{V_{\alpha}}(\mathcal{S})$. We assume that $R_v \in \mathcal{T}(X)$ and we need to show that $R_t \in \mathcal{T}(X)$ for $t = v \circ \amalg_{\alpha} q(\alpha)$.

Take therefore arbitrary $B \in \mathbf{a}$ and $f \in R_v(B)$. We have $f = \sum_{\alpha} v_{\alpha} \circ f_{\alpha}$ for certain $f_{\alpha} : B \to V_{\alpha}$. We have $\cap_{\alpha} f_{\alpha}^{-1} R_{q(\alpha)} \subset f^{-1} R_t$. Since $f_{\alpha} = 0$ for all but finitely many α , by (T2) and (T5) we find $\cap_{\alpha} f_{\alpha}^{-1} R_{q(\alpha)} \in \mathcal{T}(B)$, so by (T4) $f^{-1} R_t \in \mathcal{T}(B)$. That $R_t \in \mathcal{T}(X)$ thus follows from (T3).

Lemma 2.2.7. Consider an exact cocontinuous functor Θ : $\mathsf{PSha} \to \mathbf{C}$ to a cocomplete abelian category \mathbf{C} and set $u := \Theta \circ \mathbf{Y} : \mathbf{a} \to \mathbf{C}$.

- (i) The class $\mathcal{C}o(\mathcal{S}(\Theta)) = \widetilde{\mathcal{C}o}(\mathcal{S}(\Theta))$ comprises all morphisms $\amalg_{\beta}Y_{\beta} \to X$ for which the induced morphism $\oplus_{\beta}u(Y_{\beta}) \to u(X)$ is an epimorphism.
- (ii) The class $\mathcal{S}(\Theta)$ is a pretopology on **a**.

Proof. Part (i) follows from the fact that $\mathbf{a} \subset \mathsf{PSha}$ is a generator, and the assumptions on Θ .

Now we prove part (ii). Consider the solid diagram in (PTa). Let P denote the pullback in PSha of q and f. Then P is a quotient of some $\bigoplus_{\delta} \mathbb{Y}(C_{\delta})$. Composing the morphisms yields a commutative diagram as requested where it remains to be shown that the induced composite

$$\bigoplus_{\delta} u(C_{\delta}) \to \Theta(P) \to u(A)$$

is an epimorphism. That the left morphism is an epimorphism follows from right exactness of Θ . Furthermore, left exactness of Θ implies that $\bigoplus_{\beta} u(Y_{\beta}) \leftarrow \Theta(P) \rightarrow u(A)$ is a pullback. That $\Theta(P) \rightarrow u(A)$ is an epimorphism therefore follows from the fact that u(q) was an epimorphism and [Fr, Pullback Thm 2.54]. We can prove similarly that (PTb) is satisfied. \Box

Proof of Theorem 2.2.5. Part (i) for $\operatorname{pre}(\mathcal{T})$ is a special case of Lemma 2.2.7(ii). That (PTa) is satisfied for $\operatorname{pre}'(\mathcal{T})$ follows immediately from (T2). That (PTb') is satisfied follows from the fact that the exactness in (PTb') is just a reformulation of exactness in PSha.

Now we prove that $\operatorname{top}(\mathcal{S})$ is a topology. Condition (T1) follows by definition. Now take $R \in \operatorname{top}(\mathcal{S})(X)$. By definition, we have $R_r \subset R$ for some $r \in \widetilde{\mathcal{C}o}_X(\mathcal{S})$. Using (PTc) allows us to conclude that for $f : A \to X$ we have $R_{r'} \subset f^{-1}R$ for some $r' \in \widetilde{\mathcal{C}o}_A(\mathcal{S})$. Hence (T2) follows. Finally, consider sieves R, S on X as in (T3). By assumption, we have $R_r \subset R$ for some $r : \amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{\mathcal{C}o}(\mathcal{S})$. We should also have $R_{t(\alpha)} \subset r_{\alpha}^{-1}S$ for some $t(\alpha) \in \widetilde{\mathcal{C}o}_{V_{\alpha}}(\mathcal{S})$, for every α . But this means that $R_{r_{\alpha} \circ t(\alpha)} \subset S$ for every α , and consequently $R_s \subset S$ for $s = r \circ (\amalg_{\alpha} t(\alpha))$. Hence S is in top(\mathcal{S}). This concludes the proof of part (ii).

Part (iii) follows immediately from the definitions and property (T4) of \mathcal{T} . Part (iv) is immediate by construction.

Finally, we prove part (v). By (iv) we have $\mathcal{T}_j \subset \operatorname{top}(\cup_i \mathcal{S}_i)$, for every $j \in I$. Now assume that $\mathcal{T}_j \subset \mathcal{T}$ for every j. In particular, $R_q \in \mathcal{T}$ for every $q \in \cup_i \mathcal{C}o(\mathcal{S}_i) = \mathcal{C}o(\cup_i \mathcal{S}_i)$. By Lemma 2.2.6, we thus find $\operatorname{top}(\cup_i \mathcal{S}_i) \subset \mathcal{T}$, which concludes the proof.

Corollary 2.2.8. (i) For pretopologies S_1 and S_2 on \mathbf{a} , we have $\operatorname{top}(S_2) \subset \operatorname{top}(S_1)$ if and only if for each $\amalg_{\beta} Y_{\beta} \to X$ in $\operatorname{Co}(S_2)$, there exist $r \in \widetilde{\operatorname{Co}}_X(S_1)$ with a commutative diagram



KEVIN COULEMBIER

(ii) Consider a topology \mathcal{T} and a pretopology \mathcal{S} on \mathbf{a} consisting of sequences which become right exact after application of $Z : \mathbf{a} \to \mathsf{Sh}(\mathbf{a}, \mathcal{T})$. Then $\operatorname{top}(\mathcal{S}) = \mathcal{T}$ if and only if for each formal morphism $\amalg_{\beta} Y_{\beta} \to X$ for which $\bigoplus_{\beta} Z(Y_{\beta}) \to Z(X)$ is an epimorphism, there exist $r \in \widetilde{Co}_X(\mathcal{S})$ with a commutative diagram as in (i).

Proof. Part (i) is a special case of Lemma 2.2.6. For part (ii), we have by assumption $\mathcal{S} \subset \operatorname{pre}\mathcal{T}$, so $\operatorname{top}\mathcal{S} \subset \mathcal{T}$ by Theorem 2.2.5. It thus suffices to prove that $\mathcal{T} \subset \operatorname{top}\mathcal{S}$ is equivalent to the condition in (ii). By Lemma 2.2.7(i), this is the special case of part (i) for $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}_2 = \operatorname{pre}(\mathcal{T})$.

Remark 2.2.9. When comparing Definition 2.2.1 with the classical notion of a pretopology in Definition 1.5.1, in some sense 2.2.1 is much broader. As made more precise in Subsection 2.4, unenriched pretopologies are closer to classes S of sequences (4) which satisfy

- $Co(S) = \widetilde{Co}(S)$ as an analogue of (pt1), (pt3);
- (PTa) as an analogue of (pt2);
- (PTb') as an analogue of the universal property of fibre products.

Even though these are satisfied by $\operatorname{pre}'(\mathcal{T})$ for every additive topology \mathcal{T} , we keep the definition broader, for instance to have that unions of pretopologies are again pretopologies.

Example 2.2.10. For a commutative k-algebra K we will denote formal sequences in its one object category as sequences in Mod_K . For a set $\mathbf{x} = \{x_\alpha \in K\}$, consider the sequence

$$s_{\mathbf{x}} : \bigoplus_{\alpha \neq \beta} K \to \bigoplus_{\alpha} K \to K$$

where the right morphisms come from $1 \mapsto x_{\alpha}$ and the left morphisms send 1 in the (α, β) labelled copy of K to 0 everywhere, except to x_{β} in the α -copy of K and to $-x_{\alpha}$ in the β -copy. For any collection E of such sets \mathbf{x} , the collection $\{s_{\mathbf{x}} \mid \mathbf{x} \in E\}$ is a pretopology. This gives a unified construction of a pretopology for every Gabriel topology on K.

- (i) If we take a set E of elements in K, then top $(\{s_{\{x\}}|x \in E\})$ is the topology corresponding to the localisation Mod_{K_E} of Mod_K .
- (ii) Let k be a field, and K = k[x, y]. Then top($\{s_{\{x,y\}}\}$) is the topology corresponding to the localisation QCohX of $Mod_{k[x,y]}$ with $X = \mathbb{A}^2 \setminus \{0\}$.

2.3. Sheaves.

Theorem 2.3.1. For a pretopology S, set T := top(S). Then $Sh_S a = Sh(a, top(S))$. So in particular, $Sh_S a$ is a Grothendieck category.

We will write the proof of the theorem in a couple of steps. Throughout, we keep the pretopology S fixed and always set $\mathcal{T} = top(S)$.

For a morphism $v : \amalg_{\alpha} V_{\alpha} \to X$ we can complete $\bigoplus_{\alpha} \mathbb{Y}(V_{\alpha}) \twoheadrightarrow R_{v}$ to a presentation of R_{v} in PSha, such that applying Nat(-, F), for $F \in \mathsf{PSha}$, yields the exact sequence

$$0 \to \operatorname{Nat}(R_v, F) \to \prod_{\alpha} F(V_{\alpha}) \to \prod_{f: W \to \amalg_{\alpha} V_{\alpha} \mid v \circ f = 0} F(W).$$
(6)

Lemma 2.3.2. The following conditions are equivalent for $F \in \mathsf{PSha}$.

- (a) F is \mathcal{T} -separated.
- (b) $F(X) \to \prod_{\alpha} F(V_{\alpha})$ is a monomorphism for every $\amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{Co}(\mathcal{S})$.

(c) $F(X) \to \prod_{\beta} F(Y_{\beta})$ is a monomorphism for every $\amalg_{\beta} Y_{\beta} \to X$ in Co(S).

Proof. Since compositions of monomorphisms are monomorphisms, it is clear that (b) and (c) are equivalent. We thus focus on the equivalence between (a) and (b).

It follows from the definition of $top(\mathcal{S})$ that F is separated if and only if $F(X) \rightarrow Nat(R_s, F)$ is a monomorphism for every $s \in \widetilde{Co}(\mathcal{S})$. The presentation (6) shows that the latter is indeed equivalent with condition (b).

Lemma 2.3.3. Let F be a \mathcal{T} -separated presheaf.

(i) For every sequence (4) in S, we have

$$\ker\left(\prod_{\beta} F(Y_{\beta}) \to \prod_{f:U \to \amalg_{\beta} Y_{\beta} \mid q \circ f = 0} F(U)\right) = \ker\left(\prod_{\beta} F(Y_{\beta}) \to \prod_{\gamma} F(Z_{\gamma})\right).$$

(ii) Consider $v : \amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{Co}(\mathcal{S})$ and a morphism $q(\alpha) : \amalg_{\beta} W_{\beta}(\alpha) \to V_{\alpha}$ in $Co(\mathcal{S})$, for each α , and set $w = v \circ \amalg_{\alpha} q(\alpha)$. The kernel of

$$\prod_{\alpha} F(V_{\alpha}) \rightarrow \prod_{f: U \to \amalg_{\alpha} V_{\alpha} \mid v \circ f = 0} F(U)$$

equals the kernel of the composite of

$$\prod_{\alpha} F(V_{\alpha}) \rightarrow \prod_{\alpha} \prod_{\beta} F(W_{\beta}(\alpha)) \rightarrow \prod_{g: Q \rightarrow \amalg_{\alpha} \amalg_{\beta} W_{\beta}(\alpha) \mid w \circ g = 0} F(Q).$$

(iii) For $r : \amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{\mathcal{Co}}(\mathcal{S})$ and any sieve R on X containing R_r , the inclusion $R_r \subset R$ induces a monomorphism $\operatorname{Nat}(R, F) \to \operatorname{Nat}(R_r, F)$.

Proof. For both proposed equalities in (i) and (ii), inclusion in one direction is obvious, so we only prove inclusion in the other direction.

For part (i), take $y \in \prod_{\beta} F(Y_{\beta})$ which is sent to zero in $\prod_{\gamma} F(Z_{\gamma})$. We need to show it is also in the kernel on the left-hand side. So consider $f : U \to \coprod_{\beta} Y_{\beta}$ with $q \circ f = 0$. By assumption (PTb) there exists a commutative diagram with $q' : \amalg_{\alpha} B_{\alpha} \to U$ in $\widetilde{Co}(S)$ such that evaluation of F yields a commutative diagram



The lower horizontal arrow is a monomorphism by assumption that F be separated, see Lemma 2.3.2. Hence y is indeed sent to zero in F(U) as desired.

Now we prove part (ii). Consider $x \in \prod_{\alpha} F(V_{\alpha})$ sent to zero by the composite morphism. We need to show it is also in the first kernel. Consider first an arbitrary $f: U \to \amalg_{\alpha} V_{\alpha}$. By Lemma 2.2.3(i) there exists some $q': \amalg_{\delta} C_{\delta} \to U$ in $\widetilde{Co}(\mathcal{S})$, such that evaluation of F yields a commutative diagram



Now assume that $v \circ f = 0$. It then follows from that x is sent to zero by the upper path from top right to bottom left in the diagram. Consequently $x \mapsto 0$ under $\prod_{\alpha} F(V_{\alpha}) \to F(U)$. This concludes the proof of part (ii).

Now we prove part (iii). Consider arbitrary $A \in \mathbf{a}$ and $f \in R(A) \subset \mathbf{a}(A, X)$. By (PTc) we have an $r' : \amalg_{\beta} B_{\beta} \to A$ in $\widetilde{Co}(S)$ yielding a commutative diagram in PSha



Applying Nat(-, F) and the Yoneda lemma yields the following commutative diagram:

If $\eta \in \operatorname{Nat}(R, F)$ is sent to 0 in $\operatorname{Nat}(R_r, F)$, it thus follows that $\eta_A(f) = 0$. Since $f \in R(A)$ was arbitrary, $\eta_A : R(A) \to F(A)$ is zero. But also A was arbitrary, so $\eta = 0$.

Corollary 2.3.4. A \mathcal{T} -separated presheaf F is a \mathcal{T} -sheaf if and only if the sequence

$$F(X) \to \prod_{\alpha} F(V_{\alpha}) \to \prod_{f:W \to \amalg_{\alpha} V_{\alpha} \mid r \circ f = 0} F(W)$$

is exact for every $r: \amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{\mathcal{Co}}(\mathcal{S})$.

Proof. By Lemma 2.3.2 and exactness in (6), exactness of the displayed sequence is equivalent with $F(X) \to \operatorname{Nat}(R_r, F)$ being an isomorphism, for every $r \in \widetilde{Co}_X(\mathcal{S})$. In particular, the sequence is exact for a sheaf. Now for any $R \in \mathcal{T}(X)$ we have $R_r \subset R \subset \mathbf{a}(-, X)$ for some $r \in \widetilde{Co}(\mathcal{S})$. By Lemma 2.3.3(iii), the inclusions yield monomorphisms

$$F(X) \hookrightarrow \operatorname{Nat}(R, F) \hookrightarrow \operatorname{Nat}(R_r, F).$$

Consequently, if the composite is an isomorphism, then so is the left arrow. This proves the second direction of the claim. $\hfill \Box$

Proof of Theorem 2.3.1. Based on Lemma 2.3.2 and Corollary 2.3.4 it now suffices to prove that the following properties are equivalent for a separated presheaf F:

(p) The sequence

$$F(X) \rightarrow \prod_{\beta} F(Y_{\beta}) \rightarrow \prod_{\gamma} F(Z_{\gamma})$$

is exact for every $\amalg_{\gamma} Z_{\gamma} \to \amalg_{\beta} Y_{\beta} \to X$ in \mathcal{S} .

(t) The sequence

$$F(X) \to \prod_{\alpha} F(V_{\alpha}) \to \prod_{f:U \to \amalg_{\alpha} V_{\alpha} \mid r \circ f = 0} F(U)$$

is exact for every $r: \amalg_{\alpha} V_{\alpha} \to X$ in $\widetilde{\mathcal{Co}}(\mathcal{S})$.

If (t) is satisfied, then considering the special case $r \in Co(S)$ together with Lemma 2.3.3(i) shows that (p) is also satisfied.

Now assume that (p) is satisfied for a separated presheaf F. We will prove by induction on $i \in \mathbb{N}$ that (t) is satisfied for $r \in Co^i(S)$. For i = 0 the statement is trivial and for i = 1 it is immediate (or follows from 2.3.3(i)).

Assume therefore that for some $r : \amalg_{\alpha} V_{\alpha} \to X$ in $\mathcal{C}o^{i}(\mathcal{S})$, the sequence in (t) is exact. Now take for each α a sequence

$$\amalg_{\gamma} Z_{\gamma}(\alpha) \xrightarrow{p(\alpha)} \amalg_{\beta} W_{\beta}(\alpha) \xrightarrow{q(\alpha)} V_{\alpha}$$

in \mathcal{S} and consider $w = r \circ \amalg_{\alpha} q(\alpha) \in \mathcal{C}o^{i+1}(\mathcal{S})$. We need to prove that

$$F(X) \to \prod_{\alpha} \prod_{\beta} F(W_{\beta}(\alpha)) \to \prod_{g: Q \to \amalg_{\alpha} \amalg_{\beta} W_{\beta}(\alpha) \mid w \circ g = 0} F(Q)$$

is exact. We will exploit that the left morphism factors via $F(X) \to \prod_{\alpha} F(V_{\alpha})$ to prove exactness in two steps.

Take $m \in \prod_{\alpha} \prod_{\beta} F(W_{\beta}(\alpha))$ such that $m \mapsto 0$ in the above sequence. By letting g range over all morphisms of the composite form $Z_{\gamma_0}(\alpha_0) \to \amalg_{\beta} W_{\beta}(\alpha_0) \to \amalg_{\alpha} \amalg_{\beta} W_{\beta}(\alpha)$ and using assumption (p) for each α_0 , we find that m is the image of some n under

 $\prod_{\alpha} F(V_{\alpha}) \to \prod_{\alpha} \prod_{\beta} F(W_{\beta}(\alpha))$. That *n* is in the image of $F(X) \to \prod_{\alpha} F(V_{\alpha})$ then follows from Lemma 2.3.3(ii) and our assumption on *r*.

We conclude the subsection with some immediate consequences of Theorems 2.3.1 and 1.4.5.

Corollary 2.3.5. For a class S of sequences (4), all the induced sequences

$$\bigoplus_{\gamma} \mathsf{Z}(Z_{\gamma}) \to \bigoplus_{\beta} \mathsf{Z}(Y_{\beta}) \to \mathsf{Z}(X) \to 0$$

are exact in $Sh_{\mathcal{S}}a$. In particular, we have $\mathcal{S} \subset pre(top\mathcal{S})$ for a pretopology \mathcal{S} .

Corollary 2.3.6. Consider a creator $u : \mathbf{a} \to \mathbf{C}$ of a Grothendieck category \mathbf{C} . Then \mathbf{C} is equivalent to the full subcategory of $F \in \mathsf{PSha}$ for which

$$0 \to F(X) \to \prod_{\beta} F(Y_{\beta}) \to \prod_{\gamma} F(Z_{\gamma})$$

is exact in Mod_k for every sequence (4) for which the following is exact in C:

$$\bigoplus_{\gamma} u(Z_{\gamma}) \to \bigoplus_{\beta} u(Y_{\beta}) \to u(X) \to 0.$$

Corollary 2.3.7. [LRS, Proposition 2.8] The intersection of a family of Giraud subcategories of PSha is again Giraud. Concretely, for a family of topologies $\{\mathcal{T}_i | i \in I\}$ on \mathbf{a} , we have

$$\bigcap_{i\in I} \mathsf{Sh}(\mathbf{a},\mathcal{T}_i) = \mathsf{Sh}(\mathbf{a},\vee_{i\in I}\mathcal{T}_i).$$

Proof. By definition, for pretopologies S_i , the intersection of the subcategories $\mathsf{Sh}_{S_i}\mathbf{a}$ equals $\mathsf{Sh}_{\cup S_i}\mathbf{a}$. The conclusion thus follows from Theorems 2.3.1 and 2.2.5(v).

Remark 2.3.8. The condition on a class S of sequences (4) to be a pretopology is not necessary for Sh_Sa to be a Giraud subcategory of PSha. There are examples where $Sh_Sa = 0$ while S is not a pretopology.

2.4. Linearising unenriched pretopologies. For an essentially small category **c** which is not k-linear, we can consider the k-linear category $k\mathbf{c}$. We have $Obk\mathbf{c} = Ob\mathbf{c}$ and $k\mathbf{c}(X, Y)$ is the free k-module on generators $\mathbf{c}(X, Y)$.

2.4.1. Consider a (unenriched) Grothendieck pretopology on **c**. For each covering $\{U_i \rightarrow U\}$ in the pretopology, we consider the formal sequence

$$\amalg_{i,l}U_i \times_U U_l \to \amalg_i U_i \to U$$

in $k\mathbf{c}$ (with fibre product to be interpreted in \mathbf{c}). The morphism $c = (c_i) : U_j \times_U U_l \to \amalg_i U_i$ is such that c_j is the projection $U_j \times_U U_l \to U_j$, while c_l is -1 times the projection $U_j \times_U U_l \to U_l$ and $c_i = 0$ when $i \notin \{j, l\}$.

Denote the class of all sequences constructed this way by \mathcal{S} .

Theorem 2.4.2. The class S from 2.4.1 is a k-linear Grothendieck pretopology on kc.

Proof. We start by proving (PTa). Consider a covering $\{U_i \to U\}$ and a morphism $f \in k\mathbf{c}(A, U)$. If $f = \lambda f_0$ for $f_0 \in \mathbf{c}(A, U)$ and $\lambda \in k$, then the condition in (PTa) follows easily from (pt2) with $\sqcup_{\delta} C_{\delta} := \amalg_i U_i \times_U A$ where the fibre products are taken with respect to f_0 . Condition (PTa) for arbitrary morphisms $f = \sum_i \lambda_i f_i$ with $\lambda_i \in k$ and $f_i \in \mathbf{c}(A, U)$ then follows from an iterative argument almost identical to the one in the proof of Lemma 2.2.3(i).

We claim that (PTb'), so in particular (PTb), holds. Indeed, this follows from exactness in (3) and the fact that the functor $\mathsf{Set} \to \mathsf{Mod}_k$, which sends a set S to the k-module kS it freely generates, sends exact diagrams $A \Rightarrow B \to C$ to acyclic sequences $kA \to kB \to kC$. \Box

Remark 2.4.3. In [AGV, II.1.4], the (unenriched) topology associated to a pretopology on \mathbf{c} is described. We can take the canonical linearisation of that topology, which is the k-linear topology on $k\mathbf{c}$ comprising all sieves which contain a trivial linearisation of a sieve in the unenriched topology. The latter equals the k-linear topology top \mathcal{S} , with \mathcal{S} as in 2.4.1.

2.5. Universal property. Fix k-linear cocomplete categories **B** and **C**. Denote the category of k-linear cocontinuous functors $\mathbf{B} \to \mathbf{C}$ by $[\mathbf{B}, \mathbf{C}]_{cc}$.

2.5.1. For a class S of sequences (4) in **a**, we denote by $[\mathbf{a}, \mathbf{B}]_S$ the category of k-linear functors $h : \mathbf{a} \to \mathbf{B}$ for which

$$\bigoplus_{\gamma} h(Z_{\gamma}) \to \bigoplus_{\beta} h(Y_{\beta}) \to h(X) \to 0$$

is exact (is a cokernel diagram) in \mathbf{B} , for every sequence (4) in \mathcal{S} .

The following type of result is standard, see for instance [Ke, §6.4] or [Sc2, §3].

Proposition 2.5.2. Let S be a class of sequences (4) for which $Sh_S a \rightarrow PSha$ is reflective (for instance S is κ -bounded or a pretopology). Then precomposition with $Z : a \rightarrow Sh_S a$ yields an equivalence

$$-\circ \mathbf{Z}: [\mathbf{Sh}_{\mathcal{S}}\mathbf{a}, \mathbf{B}]_{cc} \xrightarrow{\sim} [\mathbf{a}, \mathbf{B}]_{\mathcal{S}}.$$

We start the proof with the following lemma.

Lemma 2.5.3. Consider $u \in [\mathbf{a}, \mathbf{C}]$ for which the right adjoint $R : \mathbf{C} \to \mathsf{PSha}, M \mapsto \mathbf{C}(u-, M)$ is fully faithful.

(i) For an object $M \in \mathbf{C}$, consider the category \mathbf{j}_M with as object pairs (A, f) with $A \in \mathbf{a}$ and $f: u(A) \to M$, and where a morphism $(A, f) \to (B, g)$ is a morphism $a: A \to B$ in \mathbf{a} such that u(a), f, g yield a commutative diagram. For the obvious forgetful functor $J_M: \mathbf{j}_M \to \mathbf{a}$, we have a canonical isomorphism

$$\operatorname{colim}(u \circ J_M) \xrightarrow{\sim} M.$$

(ii) Consider k-linear cocontinuous functors $H_1, H_2 : \mathbf{C} \to \mathbf{B}$ and a natural transformation $\eta : H_1 \to H_2$. If η yields an isomorphism $H_1 \circ u \xrightarrow{\sim} H_2 \circ u$, then η is an isomorphism too.

Proof. Clearly, \mathbf{j}_M is equivalent to the category $\mathbf{el} := \mathbf{el}(RM)$ of elements of the functor $RM : \mathbf{a}^{\mathrm{op}} \to \mathsf{Mod}_k$, that is the category of pairs (A, p) with $A \in \mathbf{a}$ and $p \in (RM)(A)$. For the forgetful $J : \mathbf{el} \to \mathbf{a}$, we have $\operatorname{colim}(\mathbf{Y} \circ J) \xrightarrow{\sim} RM$. Part (i) then follows from the isomorphisms, natural in $N \in \mathbf{C}$:

$$\mathbf{C}(M,N) \xrightarrow{\sim} \mathsf{PSha}(RM,RN) \xrightarrow{\sim} \lim \mathsf{PSha}(\mathsf{Y} \circ J,RN) \xrightarrow{\sim} \lim \mathsf{PSha}(\mathsf{Y} \circ J_M,RN)$$

$$\xrightarrow{\sim} \lim \mathbf{C}(u \circ J_M,N) \xrightarrow{\sim} \mathbf{C}(\operatorname{colim} u \circ J_M,N).$$

Part (ii) is an immediate application of part (i) by replacing every object M in \mathbb{C} by its colimit over \mathbf{j}_M .

Proof of Proposition 2.5.2. For a reflective subcategory **C** of PSha, the inclusion $I : \mathbf{C} \rightarrow P$ Sha is the right adjoint of $Z : \mathbf{a} \rightarrow \mathbf{C}$ (more precisely of **S**). We can thus apply Lemma 2.5.3.

Since **a** is essentially small and **B** is cocomplete, for any k-linear functor $h : \mathbf{a} \to \mathbf{B}$, we have the left Kan extension

$$L(h) \coloneqq \operatorname{Lan}_{\mathsf{Z}}(h) : \operatorname{Sh}_{\mathcal{S}} \mathbf{a} \to \mathbf{B}$$

along Z, see [Ma, Chapter X] or [Ke, Chapter 4]. By definition, there exists a natural transformation $h \to L(h) \circ Z$ which induces an isomorphism

$$\operatorname{Nat}(L(h), G) \simeq \operatorname{Nat}(h, G \circ Z),$$

for any k-linear functor $G: \mathsf{Sh}_{\mathcal{S}}\mathbf{a} \to \mathbf{B}$. In other words, we have an adjunction

$$[\mathbf{a}, \mathbf{B}] \underbrace{\overset{L(-)}{\longleftarrow}}_{-\circ \mathbf{Z}} [\mathsf{Sh}_{\mathcal{S}} \mathbf{a}, \mathbf{B}], \tag{7}$$

and we will show it restricts to an equivalence on the requested subcategories.

For a sequence (4) in \mathcal{S} , the sequence

$$\bigoplus_{\gamma} \mathsf{Z}(Z_{\gamma}) \to \bigoplus_{\beta} \mathsf{Z}(Y_{\beta}) \to \mathsf{Z}(X) \to 0$$

is exact in $\mathsf{Sh}_{\mathcal{S}}\mathbf{a}$, by Corollary 2.3.5. Since $H \in [\mathsf{Sh}_{\mathcal{S}}\mathbf{a}, \mathbf{B}]_{cc}$ is right exact, it maps the above exact sequence to an exact one. Since H commutes with coproducts, it follows that $h := H \circ \mathbb{Z}$ is in $[\mathbf{a}, \mathbf{B}]_{\mathcal{S}}$, which means $-\circ \mathbb{Z}$ restricts indeed to a functor $[\mathsf{Sh}_{\mathcal{S}}\mathbf{a}, \mathbf{B}]_{cc} \to [\mathbf{a}, \mathbf{B}]_{\mathcal{S}}$.

Now assume that $h \in [\mathbf{a}, \mathbf{B}]_{\mathcal{S}}$, then we have the functor

$$h: \mathbf{B} \to \mathsf{Sh}_{\mathcal{S}}\mathbf{a}, \ M \mapsto \mathbf{B}(h-, M).$$

We claim that $(L(h), \hat{h})$ is an adjoint pair, which shows in particular that L(h) is cocontinuous. Indeed, this standard property can be derived from the coend expression

$$\operatorname{Lan}_{\mathbf{Z}}h(F) = \int^{X \in \mathbf{a}} \operatorname{Sh}_{\mathcal{S}}\mathbf{a}(\mathbf{Z}X, F) \odot_k h(X) \simeq \int^{X \in \mathbf{a}} F(X) \odot_k h(X),$$

for $F \in \mathsf{Sh}_{\mathcal{S}}\mathbf{a}$. In the above formula, $N \odot_k B$ for $N \in \mathsf{Mod}_k$ and $B \in \mathbf{B}$ stands for the object in **B** with the universal property $\mathbf{B}(N \odot_k B, -) \simeq \operatorname{Hom}_k(N, \mathbf{B}(B, -))$.

Hence (7) restricts to an adjunction between $[\mathbf{a}, \mathbf{B}]_{\mathcal{S}}$ and $[\mathsf{Sh}_{\mathcal{S}}\mathbf{a}, \mathbf{B}]_{cc}$. Using the adjoint pairs $(L(h), \hat{h})$ and (\mathbf{S}, \mathbf{I}) , we find isomorphisms

 $\mathbf{B}(L(h)(\mathbf{Z}X), M) \simeq \mathsf{Sh}_{\mathcal{S}}\mathbf{a}(\mathbf{Z}X, \hat{h}M) \simeq \operatorname{Nat}(\mathbf{a}(-, X), \hat{h}M) \simeq \mathbf{B}(hX, M),$

natural in $X \in \mathbf{a}$ and $M \in \mathbf{B}$. This shows that the unit of $L(-) \dashv -\circ \mathbf{Z}$ induces an isomorphism $h \simeq L(h) \circ \mathbf{Z}$.

To prove that L(-) and $-\circ Z$ are inverses, it is now enough to show that the right adjoint, $-\circ Z$, reflects isomorphisms. The latter is precisely Lemma 2.5.3(ii).

Remark 2.5.4. Consider a cocontinuous functor $H : \mathsf{Sh}(\mathbf{a}, \mathcal{T}) \to \mathbf{B}$. Define $h \coloneqq H \circ \mathsf{Z}$. The above proof shows that $H \simeq \operatorname{Lan}_{\mathsf{Z}} h$ and its right adjoint is given by $M \mapsto \mathbf{B}(h^-, M)$.

2.5.5. Denote by $\mathsf{Top}(\mathbf{a})$ the partially ordered (by inclusion) set of Grothendieck topologies on **a**. We interpret $\mathsf{Top}(\mathbf{a})$ as a category and we will show that $\mathsf{Top}(\mathbf{a})$ is in fact canonically the truncation of a 2-category.

We introduce the category $Cr(\mathbf{a})$. Objects in $Cr(\mathbf{a})$ are creators $u : \mathbf{a} \to \mathbf{C}$. Morphisms $u \to u'$ are the isomorphism classes of cocontinuous k-linear functors $\mathbf{C} \to \mathbf{C}'$ which yield a commutative diagram with $\mathbf{C} \xleftarrow{u} \mathbf{a} \xrightarrow{u'} \mathbf{C}'$.

Corollary 2.5.6. The assignment $\mathcal{T} \mapsto \{ Z : \mathbf{a} \to Sh(\mathbf{a}, \mathcal{T}) \}$ yields an equivalence $\mathsf{Top}(\mathbf{a}) \xrightarrow{\sim} \mathsf{Cr}(\mathbf{a})$.

Proof. Consider two Grothendieck topologies $\mathcal{T}_1, \mathcal{T}_2$, set $\mathcal{S}_1 = \text{pre}'(\mathcal{T}_1)$ and consider $Z_i : \mathbf{a} \to \mathsf{Sh}(\mathbf{a}, \mathcal{T}_i)$. By Proposition 2.5.2, we have an equivalence

$$-\circ \mathtt{Z}_1 \colon [\mathsf{Sh}_{\mathcal{S}_1}\mathbf{a},\mathsf{Sh}(\mathbf{a},\mathcal{T}_2)] \xrightarrow{\sim} [\mathbf{a},\mathsf{Sh}(\mathbf{a},\mathcal{T}_2)]_{\mathcal{S}_1}$$

The functors F in the left category for which there exists an isomorphism $F \circ Z_1 \simeq Z_2$ are by definition the ones which are mapped (up to isomorphism) to Z_2 . Such functors thus exist (and are then all isomorphic) if and only if Z_2 maps all sequences in S_1 to right exact sequences in $Sh(\mathbf{a}, \mathcal{T}_2)$. That condition is a reformulation of $\operatorname{pre}'(\mathcal{T}_1) \subset \operatorname{pre}(\mathcal{T}_2)$, which we claim is equivalent with $\mathcal{T}_1 \subset \mathcal{T}_2$. Indeed, by Theorem 2.2.5(iii) and (iv), $\operatorname{pre}'(\mathcal{T}_1) \subset \operatorname{pre}(\mathcal{T}_2)$ implies $\mathcal{T}_1 \subset \mathcal{T}_2$. Also by Theorem 2.2.5(iv), $\mathcal{T}_1 \subset \mathcal{T}_2$ implies $\operatorname{pre}'(\mathcal{T}_1) \subset \operatorname{pre}'(\mathcal{T}_2)$. However, we have $\operatorname{pre}'(\mathcal{T}_2) \subset \operatorname{pre}(\mathcal{T}_2)$ as a consequence of Theorem 2.2.5(iii) and Corollary 2.3.5. This demonstrates we can extend the assignment in the corollary to a fully faithful functor $\operatorname{Top}(\mathbf{a}) \to \operatorname{Cr}(\mathbf{a})$. The functor is dense by Theorem 1.4.5.

Remark 2.5.7. One can verify that the morphisms in Cr(a) automatically correspond to *exact* functors.

3. Properties of topologies and pretopologies

3.1. Noetherian topologies. For the entire subsection, we let κ be an infinite regular cardinal.

Definition 3.1.1. A Grothendieck topology \mathcal{T} is κ -noetherian if for every $X \in \mathbf{a}$, the object Z(X) is κ -compact in $Sh(\mathbf{a}, \mathcal{T})$.

It seems difficult to characterise noetherianity directly from the the topology itself. In the following proposition we investigate the connection with the generation and presentation of the sieves in the topology. Recall from Lemma 1.3.1 that a sieve R is κ -compact in PSha if and only if it has a κ -bounded presentation by representable presheaves. Similarly, $R \subset \mathbf{a}(-, X)$ is κ -generated in PSha if $R = R_v$ for some κ -bounded $v : \amalg_{\beta} Y_{\beta} \to X$.

Proposition 3.1.2. (i) If for every $X \in \mathbf{a}$ and $R \in \mathcal{T}(X)$ there exists $\mathcal{T}(X) \ni R' \subset R$ such that R' is κ -compact in PSha, then \mathcal{T} is κ -noetherian.

- (ii) The following are equivalent:
 - (a) For every $X \in \mathbf{a}$ and $R \in \mathcal{T}(X)$ there exists $\mathcal{T}(X) \ni R' \subset R$ such that R' is κ -generated.
 - (b) For every $X \in \mathbf{a}$, the object Z(X) is κ -generated in $Sh(\mathbf{a}, \mathcal{T})$.

Proof. For part (i), we consider a functor $J : \mathbf{j} \to \mathsf{Sh}(\mathbf{a}, \mathcal{T})$ from a κ -filtered category \mathbf{j} . By cocontinuity of \mathbf{S} , the colimit of J in $\mathsf{Sh}(\mathbf{a}, \mathcal{T})$ is given by $\mathsf{S}(\operatorname{colim} \mathbf{I} \circ J)$. Then by adjunction $\mathsf{S} \to \mathsf{I}$ we have

$$\mathsf{Sh}(\mathbf{a},\mathcal{T})(\mathsf{Z}(X),\operatorname{colim} J) \simeq \operatorname{Nat}(\mathsf{Y}(X), \mathsf{I} \circ \mathsf{S}(\operatorname{colim} \mathsf{I} \circ J)).$$

Recall that $\mathbf{I} \circ \mathbf{S} \simeq \Sigma \circ \Sigma$. We can rewrite the direct limit expression for Σ in (2) by restricting to the cofinal (by assumption) subset of κ -compact sieves in the topology. Hence Σ commutes with κ -filtered colimits and $\mathbf{Z}(X)$ inherits κ -compactness from $\mathbf{Y}(X)$.

That (ii)(a) implies (ii)(b) follows precisely as in the proof of part (i), by investigating κ -filtered colimits of monomorphisms. We need to use additionally that I and Σ are left exact, see 1.4.6(i).

Now assume that Z(X) is κ -generated and consider $R \in \mathcal{T}(X)$. As an object in PSha, R admits a presentation

$$\oplus_c \mathbb{Y}(K_c) \to \oplus_b \mathbb{Y}(J_b) \to R \to 0,$$

for a collection of morphisms $J_b \to X$. By Theorem 1.4.5 and the fact that **S** is cocontinuous, the sequence

$$\oplus_c \operatorname{Z}(K_c) \to \oplus_b \operatorname{Z}(J_b) \to \operatorname{Z}(X) \to 0$$

is exact in $\mathsf{Sh}(\mathbf{a}, \mathcal{T})$. By Lemma 1.3.1(i), there exists a subset $\{Y_{\beta} \to X\}$ of $\{J_b \to X\}$ of cardinality $< \kappa$ such that the sequence obtained by replacing $\bigoplus_b \mathsf{Z}(J_b)$ by $\bigoplus_{\beta} \mathsf{Z}(Y_{\beta})$ is still exact. Let $R' \subset R \subset \mathbf{a}(-, X)$ denote the sieve generated by $\{Y_{\beta} \to X\}$. Using the reverse argument from above, it follows that $R' \in \mathcal{T}(X)$. Hence (ii)(a) follows.

It is impossible to improve 3.1.2(i) to the same form of 3.1.2(ii), as the following example shows.

Example 3.1.3. For a commutative ring K, and $x \in K$, we have the pretopology

$$\mathcal{S} = \{ 0 \to K \xrightarrow{x} K \}$$

on the one object category K, see Example 2.2.10. The corresponding Gabriel topology $\mathcal{T} \coloneqq \operatorname{top}(\mathcal{S})$ consists of all ideals in K which contain x^i for some $i \in \mathbb{N}$. By Theorem 3.1.4 below, \mathcal{T} is noetherian (see also 2.2.10(i)). However, the premise of Proposition 3.1.2(i) is not always satisfied. Indeed, let k be a field and let K be the quotient of the polynomial ring $k[x_i | i \in \mathbb{N}]$ in countably many variables by the ideal $\langle x_i x_j | i \neq j \rangle$, and set $x \coloneqq x_0 \in K$. No non-zero ideal contained in $Kx_0 \in \mathcal{T}$ is finitely presented.

Theorem 3.1.4. The following conditions are equivalent on a topology \mathcal{T} :

- (i) \mathcal{T} is κ -noetherian;
- (ii) $\mathcal{T} = \operatorname{top}(\mathcal{S})$ for a κ -bounded pretopology \mathcal{S} ;
- (iii) For every functor $J : \mathbf{j} \to \mathsf{Sh}(\mathbf{a}, \mathcal{T})$ from a κ -filtered category \mathbf{j} , the canonical morphism $\operatorname{colim}(\mathbf{I} \circ J) \to \mathbf{I}(\operatorname{colim} J)$ is an isomorphism.

We start the proof with a lemma.

Lemma 3.1.5. Consider a κ -noetherian topology \mathcal{T} . Denote by $\operatorname{pre}_{\kappa}\mathcal{T} \subset \operatorname{pre}\mathcal{T}$ the subclass of all κ -bounded sequences (κ -bounded sequences (4) which become right exact in $\mathsf{Sh}(\mathbf{a},\mathcal{T})$). Then $\operatorname{pre}_{\kappa}\mathcal{T}$ is a pretopology with $\operatorname{top}(\operatorname{pre}_{\kappa}\mathcal{T}) = \mathcal{T}$.

Proof. By applying Lemma 1.3.1(ii) for $\mathsf{Sh}(\mathbf{a}, \mathcal{T})$ to any sequence in $\operatorname{pre}(\mathcal{T})$, we can cut it down to a κ -bounded sequence which is still in $\operatorname{pre}(\mathcal{T})$.

Since $\operatorname{pre}(\mathcal{T})$ is a pretopology, it follows that for $\operatorname{pre}_{\kappa}(\mathcal{T})$ the analogues of (PTa) and (PTb) are satisfied where we only require q' to be in $\mathcal{C}o_A(\operatorname{pre}\mathcal{T}) = \widetilde{\mathcal{C}o}_A(\operatorname{pre}\mathcal{T})$. By the first paragraph, we can replace q' by an element of $\mathcal{C}o(\operatorname{pre}_{\kappa}\mathcal{T})$, which yields the requested commutative diagram. That $\operatorname{top}(\operatorname{pre}_{\kappa}(\mathcal{T})) = \mathcal{T}$ follows from Corollary 2.2.8(ii) and the first paragraph.

Proof of Theorem 3.1.4. Lemma 3.1.5 shows that (i) implies (ii). By Theorem 2.3.1, that (ii) implies (iii) is a special case of Lemma 2.1.4(i). By the same reason, that (iii) implies (i) is already observed in the proof of Lemma 2.1.4.

Corollary 3.1.6. For a family $\{\mathcal{T}_i | i \in I\}$ of κ -noetherian topologies, the topology $\lor_{i \in I} \mathcal{T}_i$ is also κ -noetherian.

Proof. By Theorem 2.2.5(v), this follows from Theorem 3.1.4

Corollary 3.1.7. Consider a κ -compact creator $u : \mathbf{a} \to \mathbf{C}$ of a Grothendieck category \mathbf{C} . Then \mathbf{C} is equivalent to the full subcategory of $F \in \mathsf{PSha}$ for which

$$0 \to F(X) \to \prod_{\beta} F(Y_{\beta}) \to \prod_{\gamma} F(Z_{\gamma})$$

is exact for every κ -bounded sequence (4) for which the following is exact:

$$\bigoplus_{\gamma} u(Z) \to \bigoplus_{\beta} u(Y_{\beta}) \to u(X) \to 0.$$

Proof. This is an immediate application of Lemma 3.1.5 and Theorem 2.3.1.

In case $\kappa = \aleph_0$ and **a** is additive, we have the following obvious simplification.

Lemma 3.1.8. Assume that a is additive.

- (i) For a topology \mathcal{T} on \mathbf{a} , the following are equivalent:
 - (a) \mathcal{T} is noetherian;
 - (b) $\mathcal{T} = \operatorname{top}(\mathcal{S})$ for a pretopology \mathcal{S} consisting of sequences $Z \to Y \to X$ in \mathbf{a} ;
 - (c) $\operatorname{pre}_2 \mathcal{T}$ is a pretopology and $\mathcal{T} = \operatorname{top}(\operatorname{pre}_2 \mathcal{T})$.
- (ii) Consider a compact creator $u : \mathbf{a} \to \mathbf{C}$ of a Grothendieck category \mathbf{C} . Then \mathbf{C} is equivalent to the category of functors $F : \mathbf{a}^{\mathrm{op}} \to \mathsf{Mod}_k$ such that $0 \to F(X) \to F(Y) \to F(Z)$ is exact for every sequence $Z \to Y \to X$ in \mathbf{a} for which $u(Z) \to u(Y) \to u(X) \to 0$ is exact in \mathbf{C}

In case u is fully faithful, Lemma 3.1.8(ii) was proved in [Sc2, Theorem 3.3.1].

3.2. Subcanonical topologies.

Definition 3.2.1. A topology \mathcal{T} is **subcanonical** if every representable presheaf is a \mathcal{T} -sheaf.

Theorem 3.2.2. The following are equivalent.

- (a) \mathcal{T} is subcanonical.
- (b) $\mathcal{T} = \operatorname{top}(\mathcal{S})$ for some pretopology \mathcal{S} comprising only right exact sequences.
- (c) If $\mathcal{T} = \operatorname{top}(\mathcal{S})$ for some pretopology \mathcal{S} , then \mathcal{S} consists of right exact sequences.
- (d) $\operatorname{pre}(\mathcal{T})$ consists of right exact sequences.

(

(e) $Z: \mathbf{a} \to Sh(\mathbf{a}, \mathcal{T})$ is fully faithful.

Proof. By Corollary 2.3.5, we have $\mathcal{S} \subset \operatorname{pre}(\mathcal{T})$ when $\mathcal{T} = \operatorname{top}(\mathcal{S})$. Hence (d) implies (c). That (c) implies (b) is obvious. That (b) implies (a) follows from Theorem 2.3.1. We have $I \circ Z \simeq Y$ when the image of Y consists of sheaves, see 1.4.6(ii), so (a) implies (e). To prove that (e) implies (d), consider a sequence (4) such that

$$\bigoplus_{\gamma} \mathsf{Z}(Z_{\gamma}) \to \bigoplus_{\beta} \mathsf{Z}(Y_{\beta}) \to \mathsf{Z}(X) \to 0$$

is exact in $\mathsf{Sh}(\mathbf{a}, \mathcal{T})$. This means in particular that application of $\mathsf{Sh}(\mathbf{a}, \mathcal{T})(-, \mathsf{Z}(A))$ yields an exact sequence for each $A \in \mathbf{a}$. Since Z is fully faithful, we find that

$$0 \to \mathbf{a}(X, A) \to \prod_{\beta} \mathbf{a}(Y_{\beta}, A) \to \prod_{\gamma} \mathbf{a}(Z_{\gamma}, A)$$

is indeed exact.

Corollary 3.2.3. For a family $\{\mathcal{T}_i | i \in I\}$ of subcanonical topologies, the topology $\lor_{i \in I} \mathcal{T}_i$ is also subcanonical.

Proof. By Theorem 2.2.5(v), this follows from Theorem 3.2.2

3.3. The canonical topology.

- **Theorem 3.3.1.** (i) There exists a unique finest subcanonical topology on \mathbf{a} , the **canon**ical topology. It is given by $\lor_i \mathcal{T}_i$, for \mathcal{T}_i ranging over all subcanonical topologies.
 - (ii) There exists a unique finest subcanonical noetherian topology \mathbf{a} , the canonical noetherian topology. It is given by $\vee_i \mathcal{T}_i$, for \mathcal{T}_i ranging over all subcanonical noetherian topologies.

Proof. It suffices to observe that the join of subcanonical (resp. noetherian) topologies is again subcanonical (resp. noetherian). This follows easily from Theorem 3.2.2 (resp. Theorem 3.1.4) and Theorem 2.2.5(v).

Example 3.3.2. If **a** is abelian, the canonical noetherian topology \mathcal{T} is the one for which $\operatorname{Inda} \simeq \operatorname{Sh}(\mathbf{a}, \mathcal{T})$.

In Section 4, we will see instances of the following generalisation of Example 3.3.2.

Theorem 3.3.3. Let \mathbf{a} be an essentially small full additive subcategory of a k-linear Grothendieck category \mathbf{C} such that \mathbf{a} is a compact generator and every compact object in \mathbf{C} is a subobject of one in \mathbf{a} . Then the following is true:

- (i) The class S of all right exact sequences $Z \to Y \to X$ in \mathbf{a} (that is sequences such that $X = \operatorname{coker}(Z \to Y)$) constitutes a pretopology on \mathbf{a} .
- (ii) The topology top(S) is the canonical noetherian topology on **a**.
- (iii) We have $\mathbf{C} \simeq \mathsf{Sh}(\mathbf{a}, \operatorname{top}(\mathcal{S}))$ and \mathbf{C} is equivalent to the category of functors $F : \mathbf{a}^{\operatorname{op}} \to \operatorname{\mathsf{Mod}}_k$, for which $F(\operatorname{coker} f) \xrightarrow{\sim} \ker F(f)$ for every morphism f which has a cokernel.

Proof. By Theorems 1.1.3 and 1.4.5, $\mathbf{C} \simeq \mathsf{Sh}(\mathbf{a}, \mathcal{T})$ for some Grothendieck topology \mathcal{T} . By Theorem 3.2.2, $\operatorname{pre}(\mathcal{T})$ consists of formal right exact sequences. Hence $\operatorname{pre}_2\mathcal{T} \subset \operatorname{pre}\mathcal{T}$ consists of right exact sequences in **a** and by Lemma 3.1.8(i), $\text{pre}_2 \mathcal{T}$ is a pretopology and \mathcal{T} = $top(pre_2\mathcal{T}).$

Part (i) thus follows if $\operatorname{pre}_2 \mathcal{T}$ comprises all right exact sequences in **a**. Consider therefore an exact sequence $Z \to Y \to X \to 0$ in **a**. We claim it is also exact in **C**. Since **C** is finitely presentable, every object is a filtered colimit of compact objects. Since filtered colimits of short exact sequences in Mod_k are exact, to prove the claim it thus suffices to prove that

$$0 \to \mathbf{C}(X, V) \to \mathbf{C}(Y, V) \to \mathbf{C}(Z, V)$$

is exact, for every compact object $V \in \mathbf{C}$. By taking a copresentation of V by objects in **a** (which exists by assumption and the fact that a quotient of two compact objects is again compact), it follows that the above sequence is exact, which proves the claim.

For part (ii), we need to show that $top(\mathcal{S})$ contains every noetherian subcanonical topology. Clearly, \mathcal{S} contains every pretopology consisting of 2-bounded right exact sequences. Since top is inclusion preserving, the claim follows from Lemma 3.1.8(i) and Theorem 3.2.2.

Part (iii) follows from the first paragraph and Theorem 2.3.1.

3.4. Some results on non-subcanonical topologies.

Proposition 3.4.1. For a pretopology S, set T = top(S). The kernel J of the functor $Z: \mathbf{a} \to Sh(\mathbf{a}, \mathcal{T})$ is given by

$$J(X, X') = \bigcup_{r: \amalg_{\alpha} V_{\alpha} \to X \in \widetilde{Co}(S)} \ker \left(\mathbf{a}(X, X') \to \prod_{\alpha} \mathbf{a}(V_{\alpha}, X') \right).$$

١

In particular, Z is faithful if and only if every $q: \amalg_{\beta}Y_{\beta} \to X$ in $\mathcal{Co}(\mathcal{S})$ is epimorphic.

Proof. Using the adjoint pair (S, I) we find that the k-module $Sh(a, \mathcal{T})(Z(X), Z(X'))$ is isomorphic to (ISa(-,X'))(X). Moreover, the corresponding composite

$$\mathbf{a}(X, X') \to (\mathbf{ISa}(-, X'))(X)$$

is precisely the double evaluation of the unit id \rightarrow IS at Y(X') and X. By 1.4.6(iii) and (iv), $\sigma_{\Sigma F} \Sigma F \to \Sigma \Sigma F$ is a monomorphism for every presheaf F. The kernel J(X, X') is therefore the kernel of

$$\mathbf{a}(X,X') \to (\Sigma \mathbf{a}(-,X'))(X) = \lim_{R \in \mathcal{T}(X)} \operatorname{Nat}(R, \mathbf{Y}(X')).$$

By definition, every $R \in \mathcal{T}(X)$ contains some $R_r \in \mathcal{T}(X)$ with $r : \sqcup V_{\alpha} \to X$ in $\mathcal{C}o(\mathcal{S})$. Moreover, by the exact sequence (6) applied to $F = \Upsilon(X')$, the kernel of $\mathbf{a}(X,X') \rightarrow \operatorname{Nat}(R_r, \Upsilon(X'))$ is the kernel of $\mathbf{a}(X,X') \rightarrow \prod_{\alpha} \mathbf{a}(V_{\alpha},X')$. This leads to the description of the kernel.

Remark 3.4.2. For the specific case $\mathcal{S} = \operatorname{pre}(\mathcal{T})$, Proposition 3.4.1 is [Lo, Lemma 3.6].

3.4.3. Let \mathcal{T} be a topology on **a** and choose a pretopology \mathcal{S} such that $\mathcal{C}o(\mathcal{S}) = \mathcal{C}o(\mathcal{S})$ and $\mathcal{T} = \operatorname{top}(\mathcal{S})$. For instance, we can take $\mathcal{S} = \operatorname{pre}(\mathcal{T})$. We then assume we have a section for $\mathcal{S} \to \mathcal{C}o(\mathcal{S})$. Concretely, this is a function **p** which associates to each $q \in \mathcal{C}o(\mathcal{S})$ a formal morphism $\mathbf{p}(q)$ which forms a sequence in S together with q. For every $X \in \mathbf{a}$, the class $\mathcal{C}o_X(\mathcal{S})$ is partially ordered, and directed due to (PTa), where $q \leq q'$ for $q: \amalg_{\beta} Y_{\beta} \to X$ and $q': \amalg_b Y'_b \to X$ if there exists a (non-unique) $f: \amalg_b Y'_b \to \amalg_\beta Y_\beta$ with $q' = q \circ f$.

Lemma 3.4.4. Keep all notation and assumptions as in 3.4.3.

(i) Let $F \in \mathsf{PSha}$ be a \mathcal{T} -separated presheaf. Then

$$(\mathsf{S}F)(X) = \varinjlim_{q \in \mathcal{C}o_X(\mathcal{S})} \ker(\prod_{\beta} F(Y_{\beta}) \xrightarrow{F(\mathsf{p}(q))} \prod_{\gamma} F(Z_{\gamma})),$$

where, for $q \leq q_1$, the morphism $F(f) : \prod_{\beta} F(Y_{\beta}) \to \prod_{b} F(Y_{b}^1)$ for any f with $q \circ f = q_1$, induces the morphism between the kernels.

(ii) Assume that every $q \in Co(S)$ is an epimorphism, then

$$\mathsf{Sh}_{\mathcal{S}}\mathbf{a}(\mathsf{Z}X,\mathsf{Z}X') = \varinjlim_{q \in \mathcal{C}_{O_X}(\mathcal{S})} \ker(\prod_{\beta} \mathbf{a}(Y_{\beta},X') \xrightarrow{-\circ \mathsf{p}(q)} \prod_{\gamma} \mathbf{a}(Z_{\gamma},X')).$$

Proof. By definition of top S it follows that for any presheaf F

$$\Sigma F(X) = \varinjlim_{q \in \mathcal{C}o_X(\mathcal{S})} \operatorname{Nat}(R_q, F).$$

Moreover, $\Sigma F \simeq SF$ if F is separated, by 1.4.6(ii) and (v).

By exact sequence (6) and Lemma 2.3.3(i), we have a commutative diagram

with exact rows when F is separated. This concludes the proof of (i).

Part (ii) is a special case of part (i), by Lemma 2.3.2.

3.5. Monoidal topologies. By a 'monoidal category' we will always understand a k-linear and monoidal category for which the tensor product is k-linear in each variable.

3.5.1. Biclosed categories. A monoidal category $(\mathbf{B}, \otimes, \mathbf{1})$ is **biclosed** if for every $X \in \mathbf{B}$ the endofunctors $X \otimes -$ and $- \otimes X$ of **B** have right adjoints $[X, -]_l$ and $[X, -]_r$. We will abbreviate 'biclosed monoidal' to 'biclosed'. If **B** is a Grothendieck category, then $(\mathbf{B}, \otimes, \mathbf{1})$ is biclosed if and only if the tensor product is cocontinuous in each variable.

3.5.2. Day convolution. Let $(\mathbf{a}, \otimes, \mathbf{1})$ be an essentially small monoidal category. Then PSha is a biclosed Grothendieck category for Day convolution, which defines the tensor product via left Kan extensions, see e.g. [IK, Sc2]. Concretely, the tensor product of $F, G \in \mathsf{PSha}$ is

$$F * G = \int^{X, Y \in \mathbf{a}} F(X) \otimes_k G(Y) \otimes_k \mathbf{a}(-, X \otimes Y)$$

and for a third $H \in \mathsf{PSha}$ we have

$$[F,H]_l = \int_{X \in \mathbf{a}} \operatorname{Hom}_k(F(X), H(X \otimes -)), \text{ or } [F,H]_l(Z) = \operatorname{Nat}(F, H(- \otimes Z)),$$

and

$$[G,H]_r = \int_{Y \in \mathbf{a}} \operatorname{Hom}_k(G(Y), H(-\otimes Y)), \text{ or } [F,H]_r(Z) = \operatorname{Nat}(G, H(Z \otimes -)).$$

The Yoneda embedding $Y : a \rightarrow \mathsf{PSha}$ is canonically monoidal and, by [IK, Theorem 5.1], composition with Y yields an equivalence

$$[\mathsf{PSha}, \mathbf{B}]^{\otimes}_{cc} \xrightarrow{\sim} [\mathbf{a}, \mathbf{B}]^{\otimes}, \tag{8}$$

for any k-linear cocomplete monoidal category with cocontinuous tensor product **B**. Here and below, for any category C of functors between monoidal categories, we denote by C^{\otimes} the category of monoidal functors for which the underlying functor lies in C.

20

3.5.3. Now consider a biclosed Grothendieck category \mathbf{C} with an essentially small full monoidal subcategory $\mathbf{c} \subset \mathbf{C}$ which is also a generator. Clearly, we can always construct such a **c**. By (8), the sheafification $\mathbf{S} : \mathsf{PShc} \to \mathbf{C}$ is monoidal. More generally, for **a** monoidal, we call a localisation \mathbf{C} of PSha , for which $\mathbf{S} : \mathsf{PSha} \to \mathbf{C}$ (or equivalently $\mathbf{Z} : \mathbf{a} \to \mathbf{C}$) has a monoidal structure, a **monoidal localisation**. Theorem 1.1.3 and the above show that all biclosed Grothendieck categories are such monoidal localisations of presheaf categories.

This motivates the notion of 'monoidal topologies' introduced below. To state the results, we need some straightforward notions. For a formal sequence σ as in (4) and $A \in \mathbf{a}$, we denote by $A \otimes \sigma$ the obvious formal sequence

$$\amalg_{\gamma}(A \otimes Z_{\gamma}) \to \amalg_{\beta}(A \otimes Y_{\beta}) \to A \otimes X,$$

and similarly for $\sigma \otimes A$. For a sieve $R \subset \mathbf{a}(-, X)$ and $A \in \mathbf{a}$, we denote by $A \otimes R$ the sieve on $A \otimes X$ for which $(A \otimes R)(V)$ contains all composites $V \to A \otimes Y \xrightarrow{A \otimes f} A \otimes X$ with $f \in R(Y)$ and $Y \in \mathbf{a}$. We define $R \otimes A$ similarly.

Theorem 3.5.4. Let $(\mathbf{a}, \otimes, \mathbf{1})$ be an essentially small monoidal category. The following conditions are equivalent on a Grothendieck topology \mathcal{T} on \mathbf{a} .

- (a) The class $\operatorname{pre}(\mathcal{T})$ is closed under the operations $A \otimes -$ and $\otimes A$, for every $A \in \mathbf{a}$.
- (b) We have $\mathcal{T} = \operatorname{pre}(\mathcal{S})$ for some pretopology \mathcal{S} closed under $A \otimes -$ and $\otimes A$, for every $A \in \mathbf{a}$.
- (c) For every $X \in \mathbf{a}$ and $R \in \mathcal{T}(X)$, we have $R \otimes A \in \mathcal{T}(X \otimes A)$ and $A \otimes R \in \mathcal{T}(A \otimes X)$.
- (d) For every $F \in Sh(\mathbf{a}, \mathcal{T})$ and $A \in \mathbf{a}$, the presheaves $F(A \otimes -)$ and $F(- \otimes A)$ are \mathcal{T} -sheaves.
- (e) There exists a (automatically essentially unique) biclosed structure on $Sh(\mathbf{a}, \mathcal{T})$ for which Z admits a monoidal structure.
- A topology which satisfies one of these conditions is **monoidal**.

Proof. We start by proving the cycle $a \Rightarrow b \Rightarrow d \Rightarrow e \Rightarrow a$. That (a) implies (b) follows from Theorem 2.2.5(iii). That (b) implies (d) is a consequence of Theorem 2.3.1.

That (d) implies (e) is an immediate consequence of Day's reflection theorem. Concretely, by [Da, Theorem 1.2(ii)] applied to the generating subcategory $\mathbf{a} \in \mathsf{PSha}$ and the localisation $\mathsf{Sh}(\mathbf{a},\mathcal{T}) \in \mathsf{PSha}$, the condition that $[\mathsf{Y}(A),F]_l$ and $[\mathsf{Y}(A),F]_r$ be \mathcal{T} -sheaves, for every \mathcal{T} sheaf F and $A \in \mathbf{a}$, implies there exists a biclosed structure on $\mathsf{Sh}(\mathbf{a},\mathcal{T})$ for which S : $\mathsf{PSha} \to \mathsf{Sh}(\mathbf{a},\mathcal{T})$ is monoidal. Such a monoidal structure must be essentially unique. By equivalence (8), uniqueness of this monoidal structure is still imposed by demanding that Z has a monoidal structure. In principle, [Da, Theorem 1.2(ii)] is only concerned with nonenriched closed symmetric monoidal categories. However, as also pointed out in [Da, §0], the methods extend trivially to enriched and biclosed monoidal categories.

That (e) implies (a) follows from the assumption that the tensor product on $Sh(a, \mathcal{T})$ is cocontinuous in each variable.

Now we prove that (e) implies (c). We can decompose $\Upsilon(A) * i$, with *i* the inclusion $R \subset \Upsilon(X)$, as follows:

$$\Upsilon(A) * i : \Upsilon(A) * R \twoheadrightarrow A \otimes R \hookrightarrow \Upsilon(A \otimes X) \xrightarrow{\sim} \Upsilon(A) * \Upsilon(X).$$
(9)

Under assumption (e), **S** is monoidal by (8). That S(Y(A) * i) is an isomorphism thus follows since S(i) is an isomorphism. Since **S** sends epimorphisms to epimorphisms, it follows that **S** sends $A \otimes R \hookrightarrow Y(A \otimes X)$ to an isomorphism. By Theorem 1.4.5, this means that $A \otimes R \in \mathcal{T}(A \otimes X)$. The same observation for $R \otimes A$ shows that (c) follows.

Finally, we prove that (c) implies (d). Let F be a \mathcal{T} -sheaf. We will only prove that $F(A \otimes X) \to \operatorname{Nat}(R, F(A \otimes -))$ is an isomorphism for all $X, A \in \mathbf{a}$ and $R \in \mathcal{T}(X)$, as the case $-\otimes A$ is done identically. By adjunction, it suffices to prove that $F(A \otimes X) \to \operatorname{Nat}(\mathbb{Y}(A) * R, F)$

is an isomorphism. The composite

$$F(A \otimes X) \rightarrow \operatorname{Nat}(\mathbb{Y}(A) \ast R, F) \hookrightarrow \operatorname{Nat}(A \otimes R, F),$$

where the monomorphism is induced from the epimorphism in (9), is an isomorphism by assumption (c). Hence the left morphism in the composite is an isomorphism too. This concludes the proof.

Theorem 3.5.5. Consider an essentially small monoidal category $(\mathbf{a}, \otimes, \mathbf{1})$ with monoidal topology \mathcal{T} and a pretoplogy (not necessarily closed under tensor product) \mathcal{S} with $\mathcal{T} = \operatorname{top}(\mathcal{S})$. Consider also a subclass $\mathcal{S}_0 \subset \mathcal{S}$ such that each $\gamma \in \mathcal{S}$ is of the form $A \otimes \gamma' \otimes B$ for some $A, B \in \mathbf{a}$ and $\gamma' \in \mathcal{S}_0$. Let \mathbf{B} be a k-linear cocomplete monoidal category with cocontinuous tensor product. Composition with the monoidal functor $\mathbf{Z} : \mathbf{a} \to \operatorname{Sh}(\mathbf{a}, \mathcal{T})$ yields an equivalence

$$[\mathsf{Sh}(\mathbf{a},\mathcal{T}),\mathbf{B}]^{\otimes}_{cc} \xrightarrow{\sim} [\mathbf{a},\mathbf{B}]^{\otimes}_{\mathcal{S}_0}$$

Proof. As the tensor product in **B** is cocontinuous, it follows that $[\mathbf{a}, \mathbf{B}]_{S_0}^{\otimes} = [\mathbf{a}, \mathbf{B}]_{S}^{\otimes}$. The statement thus becomes the monoidal version of Proposition 2.5.2 and is a standard consequence of the latter, see e.g. [Sc2, §3]. We sketch an argument below.

As Z is monoidal, the functor $[\mathsf{Sh}(\mathbf{a},\mathcal{T}),\mathbf{B}]_{cc}^{\otimes} \to [\mathbf{a},\mathbf{B}]_{S}^{\otimes}$ is well-defined and also faithfulness is inherited from 2.5.2. If for a natural transformation $\eta: F \to G$ for $F, G \in [\mathsf{Sh}(\mathbf{a},\mathcal{T}),\mathbf{B}]_{cc}^{\otimes}$, we have that $\eta(\mathbf{Z}): F \circ \mathbf{Z} \to G \circ \mathbf{Z}$ is monoidal, η is itself monoidal, by cocontinuity of F, G and Lemma 2.5.3(i). Consequently, $[\mathsf{Sh}(\mathbf{a},\mathcal{T}),\mathbf{B}]_{cc}^{\otimes} \to [\mathbf{a},\mathbf{B}]_{S}^{\otimes}$ is full. Finally, that any monoidal structure on $F \circ \mathbf{Z}$ for $F \in [\mathsf{Sh}(\mathbf{a},\mathcal{T}),\mathbf{B}]_{cc}$ extends to one on F, follows either from (8) or from Theorem 3.6.2 below, by realising a monoidal structure as a natural transformation of functors $\mathbf{a} \otimes \mathbf{a} \Rightarrow \mathbf{B}$.

When a category is braided monoidal, the conditions of being left or right closed are clearly equivalent, so we simply speak of 'closed categories'. The following theorems are immediate analogues of the above results.

Theorem 3.5.6. Let $(\mathbf{a}, \otimes, \mathbf{1}, c)$ be an essentially small braided monoidal k-linear category. A Grothendieck topology \mathcal{T} on \mathbf{a} is monoidal if and only if one of the following is satisfied.

- (a) The class $\operatorname{pre}(\mathcal{T})$ is closed under the operation $A \otimes -$, for every $A \in \mathbf{a}$.
- (b) We have $\mathcal{T} = \operatorname{pre}(\mathcal{S})$ for some pretopology \mathcal{S} closed under $A \otimes -$, for every $A \in \mathbf{a}$.
- (c) For every $X \in \mathbf{a}$ and $R \in \mathcal{T}(X)$, we have $R \otimes A \in \mathcal{T}(X \otimes A)$.
- (d) For every $F \in Sh(\mathbf{a}, \mathcal{T})$ and $A \in \mathbf{a}$, the presheaf $F(A \otimes -)$ is a \mathcal{T} -sheaf.
- (e) There exists a closed structure on Sh(a, T) for which Z admits a braided monoidal structure.

Theorem 3.5.7. Let $(\mathbf{a}, \otimes, \mathbf{1}, c)$ be an essentially small braided monoidal category and \mathbf{B} a k-linear cocomplete braided monoidal category with cocontinuous tensor product. Consider a class S_0 of formal sequences (4) in \mathbf{a} such that the class S of sequences of the form $A \otimes \gamma$ with $A \in \mathbf{a}$ and $\gamma \in S_0$ is a pretopology. Set $\mathcal{T} = \text{top}S$. Composition with the braided monoidal functor $\mathbf{Z} : \mathbf{a} \to \text{Sh}(\mathbf{a}, \mathcal{T})$ yields an equivalence

$$[\mathsf{Sh}(\mathbf{a},\mathcal{T}),\mathbf{B}]_{cc}^{\otimes,b} \xrightarrow{\sim} [\mathbf{a},\mathbf{B}]_{\mathcal{S}_0}^{\otimes,b}$$

Example 3.5.8. Let **c** be the category of open subspaces of a topological space X. It is symmetric monoidal for $U \otimes V = U \times V = U \cap V$. We can consider the usual pretopology of open coverings. The corresponding linearised topology on $k\mathbf{c}$ is monoidal, for instance by application of Remark 2.4.3 and Theorem 3.5.6(b). The corresponding localisation of

$$\mathsf{PSh}(k\mathbf{c}) \simeq \mathsf{Fun}(\mathbf{c}^{\mathrm{op}}, \mathsf{Mod}_k) \simeq \mathsf{PSh}(X; k)$$

is of course the symmetric monoidal category of sheaves of k-modules on X.

3.6. Kelly product of Grothendieck categories. We show that the Kelly tensor product of two Grothendieck categories as cocomplete categories is again a Grothendieck category. This recovers the notion of 'tensor products of Grothendieck categories' from [LRS].

3.6.1. Consider cocomplete k-linear categories \mathbf{A}, \mathbf{B} and \mathbf{C} and a bifunctor $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$, cocontinuous and k-linear in each variable. This data is the Kelly tensor product of \mathbf{A} and \mathbf{B} if, for every cocomplete k-linear category \mathbf{D} , it yields an equivalence between $[\mathbf{C}, \mathbf{D}]_{cc}$ and the category of functors $\mathbf{A} \times \mathbf{B} \to \mathbf{D}$ which are k-linear and cocontinuous in each variable. We have a canonical identification of the latter category of functors with $[\mathbf{A}, [\mathbf{B}, \mathbf{D}]_{cc}]_{cc}$. Here, and below, we freely use that colimits in functor categories are computed pointwise in the target category.

For two k-linear categories \mathbf{a}, \mathbf{b} , we denote their ordinary tensor product over k by $\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \otimes_k \mathbf{b}$. Its objects are pairs (X, Y) with $X \in \mathbf{a}$ and $Y \in \mathbf{b}$ and

$$\mathbf{a} \otimes \mathbf{b}((X,Y),(X',Y')) \coloneqq \mathbf{a}(X,X') \otimes_k \mathbf{b}(Y,Y').$$

For any k-linear category \mathbf{d} we have an equivalence

$$[\mathbf{a} \otimes \mathbf{b}, \mathbf{d}] \simeq [\mathbf{a}, [\mathbf{b}, \mathbf{d}]].$$
 (10)

The following theorem mainly recovers results from [LRS].

Theorem 3.6.2. For k-linear Grothendieck categories **A** and **B**, choose realisations as $\mathbf{A} \simeq \mathsf{Sh}_{S_1}\mathbf{a}$ and $\mathbf{B} \simeq \mathsf{Sh}_{S_2}\mathbf{b}$. Denote by $\mathbf{A} \boxtimes \mathbf{B} = \mathbf{A} \boxtimes_k \mathbf{B}$ the subcategory of $\mathsf{PSh}(\mathbf{a} \otimes \mathbf{b})$ of functors F for which the sequences

$$0 \to F(X, A) \to \prod_{\beta} F(Y_{\beta}, A) \to \prod_{\gamma} F(Z_{\gamma}, A),$$
$$0 \to F(X, A) \to \prod_{\beta} F(X, B_{\beta}) \to \prod_{\gamma} F(X, C_{\gamma})$$

are exact, for all $(X, A) \in \mathbf{a} \otimes \mathbf{b}$, all $\amalg_{\gamma} Z_{\gamma} \to \amalg_{\beta} Y_{\beta} \to X$ in S_1 and $\amalg_{\gamma} C_{\gamma} \to \amalg_{\beta} B_{\beta} \to A$ in S_2 .

- (i) $\mathbf{A} \boxtimes \mathbf{B}$ is a localisation of $\mathsf{PSh}(\mathbf{a} \otimes \mathbf{b})$, so in particular it is a Grothendieck category.
- (ii) $\mathbf{A} \boxtimes \mathbf{B}$ is the Kelly product of \mathbf{A} and \mathbf{B} , so in particular independent of our choices of (\mathbf{a}, S_1) and (\mathbf{b}, S_2) .
- (iii) If $\mathbf{a} \to \mathbf{A}$ and $\mathbf{b} \to \mathbf{B}$ are κ -compact creators for an infinite regular cardinality κ , then the same is true for $\mathbf{a} \otimes \mathbf{b} \to \mathbf{A} \boxtimes \mathbf{B}$.

Proof. Denote by S_l the class of all formal sequences $\amalg_{\gamma}(Z_{\gamma}, A) \to \amalg_{\beta}(Y_{\beta}, A) \to (X, A)$ in $\mathbf{a} \otimes \mathbf{b}$ induced from S_1 ; and by S_r the class of all sequences $\amalg_{\gamma}(X, C_{\gamma}) \to \amalg_{\beta}(X, B_{\beta}) \to (X, A)$ induced from S_2 . We also set $S = S_l \cup S_r$. The equivalence, coming from (10),

$$\mathsf{PSh}(\mathbf{a} \otimes \mathbf{b}) \simeq [\mathbf{a}^{\mathrm{op}}, \mathsf{PShb}]$$

allows us to identify $\mathsf{Sh}_{\mathcal{S}_r}(\mathbf{a} \otimes \mathbf{b})$ with $[\mathbf{a}^{\operatorname{op}}, \mathbf{B}]$. The inclusion $[\mathbf{a}^{\operatorname{op}}, \mathbf{B}] \hookrightarrow [\mathbf{a}^{\operatorname{op}}, \mathsf{PShb}]$ has left adjoint given by $F \mapsto \mathbf{S} \circ F$ for $F \in [\mathbf{a}^{\operatorname{op}}, \mathsf{PShb}]$, with \mathbf{S} the sheafification for \mathbf{B} . Since we can compute limits pointwise it follows that $[\mathbf{a}^{\operatorname{op}}, \mathbf{B}]$ is a localisation of $[\mathbf{a}^{\operatorname{op}}, \mathsf{PShb}]$. Similarly, $\mathsf{Sh}_{\mathcal{S}_r}(\mathbf{a} \otimes \mathbf{b})$ is a localisation of $\mathsf{PSh}(\mathbf{a} \otimes \mathbf{b})$. It thus follows from Corollary 2.3.7 that

$$\mathbf{A} \boxtimes \mathbf{B} = \mathsf{Sh}_{\mathcal{S}}(\mathbf{a} \otimes \mathbf{b}) = \mathsf{Sh}_{\mathcal{S}_l}(\mathbf{a} \otimes \mathbf{b}) \cap \mathsf{Sh}_{\mathcal{S}_r}(\mathbf{a} \otimes \mathbf{b})$$

is a localisation of $PSh(a \otimes b)$, proving part (i).

Let **D** be any k-linear cocomplete category. By Proposition 2.5.2 and (10), we have

$$[\mathbf{A} \boxtimes \mathbf{B}, \mathbf{D}]_{cc} \simeq [\mathbf{a} \otimes \mathbf{b}, \mathbf{D}]_{\mathcal{S}} \simeq [\mathbf{a}, [\mathbf{b}, \mathbf{D}]_{\mathcal{S}_2}]_{\mathcal{S}_1} \simeq [\mathbf{a}, [\mathbf{B}, \mathbf{D}]_{cc}]_{\mathcal{S}_1} \simeq [\mathbf{A}, [\mathbf{B}, \mathbf{D}]_{cc}]_{cc},$$

which proves part (ii). Part (iii) follows from Theorem 3.1.4 and Lemma 2.1.4(iii).

Remark 3.6.3. Using Lemma 2.2.3(i) one can show that the class S in the proof of Theorem 3.6.2 satisfies (PTa). If k is a field one can similarly show that (PTb) is satisfied and hence S is a pretopology on $\mathbf{a} \otimes \mathbf{b}$. Returning to the case of an arbitrary commutative ring k, observe that condition (PTa) is sufficient to conclude that $\mathcal{T} := \text{top}(S)$ is a topology (see proof of Theorem 2.2.5(ii)). In [LRS] it is actually proved that $\mathbf{A} \boxtimes \mathbf{B} = \text{Sh}(\mathbf{a} \otimes \mathbf{b}, \mathcal{T})$. This is only clear from our approach when k is a field.

Example 3.6.4. As observed in the proof, we have $\mathsf{PSha} \boxtimes \mathbf{B} \simeq [\mathbf{a}^{\mathrm{op}}, \mathbf{B}]$.

4. Presentations of tensor categories

For the entire section, k is a field. We denote Mod_k by Vec.

4.1. **Definitions and aims.** Part of the motivation for the previous sections comes from applications to the theory of tensor categories (in the sense of [De, EGNO]).

4.1.1. A monoidal category $(\mathbf{a}, \otimes, \mathbf{1})$ (following the convention in Subsection 3.5) is **rigid** if every object $X \in \mathbf{a}$ has a left dual X^* and a right dual *X, see [EGNO, §2.10]. We say that **a** is 'a monoidal category over k' if $k \to \mathbf{a}(\mathbf{1}, \mathbf{1})$ is an isomorphism. For the rest of the paper, we assume that **a** is an additive rigid monoidal category over k. An essentially small rigid monoidal category **T** over k is a **tensor category over** k if **T** is abelian. Henceforth, when we say 'tensor category' it is understood to mean 'tensor category over some field extension K of k' (possibly K = k) and these are considered as k-linear categories.

We will use freely that $(-)^*$ has an action on morphisms via an isomorphism

$$\mathbf{a}(X,Y) \xrightarrow{\sim} \mathbf{a}(Y^*,X^*), \quad f \mapsto f^*.$$

This sends short exact sequences to short exact sequences if **a** is abelian, see [EGNO, Proposition 4.2.9]. We refer to [Co1] for the notion of tensor ideals.

4.1.2. The ind-completion of a tensor category is a biclosed Grothendieck category, as follows from [De, §7] or Theorem 3.5.4. In this section, we investigate the possibilities to present this ind-completion via a monoidal Grothendieck topology on a rigid monoidal category. With slight abuse of notation we will refer to a monoidal functor $\mathbf{a} \rightarrow \mathbf{T}$ to a tensor category \mathbf{T} as a **rigid monoidal creator** when the composite $\mathbf{a} \rightarrow \mathsf{Ind}\mathbf{T}$ is a creator (so when $\mathsf{Ind}\mathbf{T} \rightarrow \mathsf{PSha}$ is a localisation).

We specify the explicit characterisation of creators in [Lo, Theorem 1.2] to this particular setting:

Lemma 4.1.3. A monoidal functor $u : \mathbf{a} \to \mathbf{T}$ to a tensor category \mathbf{T} is a creator if and only if the following three conditions are satisfied:

- (G) Every object in **T** is a quotient of some u(X), with $X \in \mathbf{a}$.
- (F) For every morphism $a: u(X) \to u(Y)$ there exists a morphism $q: X' \to X$ such that u(q) is an epimorphism and $a \circ u(q)$ is in the image of u.
- (FF) If u(f) = 0 for $f \in \mathbf{a}(X, Y)$, there exists a morphism $q: X' \to X$ such that u(q) is an epimorphism and $f \circ q = 0$.

Example 4.1.4. The conditions in Lemma 4.1.3 can be satisfied when **T** is a tensor category over some non-trivial field extension K/k. An example is given for **a** the category '**C**' in [Co2, §2.3.3], with char(k) = 0, and **T** the abelian envelope of [GL₀, K] from [Co2, Theorem 4.2.1] or [EHS].

Proposition 4.1.5. Assume **T** is a tensor category with full additive rigid monoidal subcategory **a** such that every object in **T** is a quotient of an object in **a**. Then Ind**T** is the localisation of PSha with respect to the canonical noetherian topology on **a**, and Ind**T** is monoidally equivalent to the category of functors $\mathbf{a}^{\text{op}} \rightarrow \text{Vec}$ which send every cokernel in **a** to the corresponding kernel in Vec. *Proof.* Clearly the subcategory $\mathbf{a} \subset \mathbf{T} \subset \mathsf{Ind}\mathbf{T}$ consists of compact objects. Moreover, by duality $(-)^*$, every compact object in $\mathsf{Ind}\mathbf{T}$ (meaning every object in \mathbf{T}) is a subobject of an object in \mathbf{a} . We can then apply Theorem 3.3.3. That the equivalence between $\mathsf{Ind}\mathbf{T}$ and the category of sheaves on \mathbf{a} is monoidal follows for instance from Theorem 3.5.5.

We also record the following basic observations for future use.

Lemma 4.1.6. Consider a rigid monoidal creator $h : \mathbf{a} \to \mathbf{T}$.

(i) For every morphism $f: X \to Y$ in **a**, there exists $g: Y \to Z$ such that $g \circ f = 0$ and

$$h(X) \xrightarrow{h(f)} h(Y) \xrightarrow{h(g)} h(Z)$$

is an exact sequence in **T**.

(ii) If I is an injective object in IndT, then there exists a functor $F : \mathbf{j} \to \mathbf{a}$, with \mathbf{j} filtered, such that $\operatorname{colim}(h \circ F) \simeq I$.

Proof. We start from a morphism $a: B \to A$ in **a**. Inside PSha, the collection of all morphisms $b: C \to B$ for which $a \circ b = 0$ is jointly epimorphic onto the kernel of a. Applying the exact and cocontinuous sheafification PSha \to IndT thus yields an exact sequence

$$\bigoplus_{b:C \to B, a \circ b = 0} h(C) \to h(B) \to h(A).$$

Since the kernel of h(a) is in **T** and thus compact, additivity of **a** implies that we can pick one morphism $b: C \to B$ for which the sequence remains exact, see Lemma 1.3.1(ii). Using the duality * on **a** then yields the sequence in (i).

For part (ii), by Lemma 2.5.3(i), it suffices to show that \mathbf{j}_I is filtered. This is an immediate application of part (i).

4.2. Morphisms to the tensor unit.

4.2.1. We denote by $\mathcal{U} = \mathcal{U}(\mathbf{a})$ the class of all non-zero morphisms $U \to \mathbf{1}$ in \mathbf{a} , and by \mathcal{U}^0 the class of all morphisms $U \to \mathbf{1}$. We consider two potential properties of such $u: U \to \mathbf{1}$:

- (Ep) The morphism $u: U \to \mathbf{1}$ is an epimorphism in **a**.
- (Ex) The following diagram is a coequaliser in **a**:

$$U \otimes U \xrightarrow[U \otimes u]{u \otimes U} U \xrightarrow{u} \mathbf{1} \ .$$

We will often regard (Ex) via the equivalent formulation that the sequence

$$\sigma_u: \quad U \otimes U \xrightarrow{u \otimes U - U \otimes u} U \xrightarrow{u} \mathbf{1} \to 0$$

is exact. We have inclusions $\mathcal{U}^{ex} \subset \mathcal{U}^{ep} \subset \mathcal{U}$, for the subclasses of morphisms which satisfy (Ex) and (Ep).

Theorem 4.2.2. (i) For $u \in U$, we have $u \in U^{ex}$ if and only if u is a strict epimorphism. (ii) If **T** is a tensor category, $\mathcal{U}(\mathbf{T}) = \mathcal{U}^{ex}(\mathbf{T})$.

Proof. For part (i), if $u \in \mathcal{U}^{ex}$ then it is a cokernel, so in particular a strict epimorphism. Now assume that $u : U \to \mathbf{1}$ is a strict epimorphism and consider $f : U \to V$ in **a** which equalises $U \otimes U \Rightarrow U$, in other words,

$$u \otimes f = f \otimes u$$

We need to show that f composes to zero with every morphism $g: X \to U$ in **a** with $u \circ g = 0$, since the latter then implies f factors via u. By the displayed equation, $u \circ g = 0$ implies

$$0 = u \otimes (f \circ g) = f \circ g \circ (u \otimes X).$$

By adjunction $u \otimes X$ is an epimorphism, so indeed $f \circ g = 0$.

KEVIN COULEMBIER

Part (ii) follows from part (i). Indeed, since **1** is known to be simple in any tensor category, $\mathcal{U} = \mathcal{U}^{ep}$. Since every epimorphism in an abelian category is strict, we have $\mathcal{U}^{ep} = \mathcal{U}^{ex}$.

Remark 4.2.3. For any quasi-coherent sheaf \mathcal{M} on a scheme \mathbb{X} with epimorphism $\mathcal{M} \twoheadrightarrow \mathcal{O}_{\mathbb{X}}$, the sequence

$$\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{M} \to \mathcal{M} \to \mathcal{O}_{\mathbb{X}} \to 0$$

is exact in QCohX. Indeed, this is clearly the case for affine schemes, since for $X = \operatorname{Spec} R$ every epimorphism $M \twoheadrightarrow R$ splits. The general claim then follows by taking the stalk at each $x \in X$.

4.2.4. For a subclass $\mathcal{V} \subset \mathcal{U}$, we denote by $\overline{\mathcal{V}} \subset \mathcal{U}^0$ the closure of \mathcal{V} under tensor products. Concretely, the elements of $\overline{\mathcal{V}}$ are the morphisms of the form

$$u_1 \otimes \cdots \otimes u_n : U_1 \otimes \cdots \otimes U_n \to \mathbf{1}$$

with $u_i \in \mathcal{V}$. We then have the sieve $\operatorname{ann}_{\mathcal{V}} = \operatorname{ann}_{\overline{\mathcal{V}}}$ on 1, defined by

 $\operatorname{ann}_{\mathcal{V}}(X) = \{ u \in \mathbf{a}(X, \mathbf{1}) \mid u^* \circ v = 0 \text{ for some } v \in \overline{\mathcal{V}} \}$

We have $\operatorname{ann}_{\mathcal{V}}(X) = 0$ if and only if $\mathcal{V} \subset \mathcal{U}^{ep}$. Note also that $\mathcal{V}_1 \subset \mathcal{V}_2$ implies $\operatorname{ann}_{\mathcal{V}_1} \subset \operatorname{ann}_{\mathcal{V}_2}$. It is well-known, see for instance [Co1, Theorem 3.1.1], that $J \mapsto J(-, \mathbf{1})$ yields an inclusion preserving bijection between tensor ideals in **a** and sieves on **1**. We denote by $J_{\mathcal{V}}$ the unique tensor ideal in **a** with $J_{\mathcal{V}}(-, \mathbf{1}) = \operatorname{ann}_{\mathcal{V}}$.

Lemma 4.2.5. Consider a monoidal creator $h : \mathbf{a} \to \mathbf{C}$ with \mathbf{C} a biclosed Grothendieck category in which $\mathbf{1}$ is finitely generated. Let $f : Y \to X$ be a morphism in \mathbf{a} for which h(f) is an epimorphism. Then there exist $u : U \to \mathbf{1}$ and $g : U \otimes X \to Y$ in \mathbf{a} such h(u) is an epimorphism and $u \otimes X = f \circ g$.

Proof. We know that $h(f \otimes X^*)$ is also an epimorphism. We consider $\operatorname{co}_X : \mathbf{1} \to X \otimes X^*$. By Lemma 2.2.7 and the fact that $\mathbf{1}$ is finitely generated, there exists $u : U \to \mathbf{1}$ in \mathbf{a} such that h(u) is an epimorphism and there exists a commutative diagram

The claim now follows by applying the adjunction $- \otimes X \dashv - \otimes X^*$.

4.3. A set of Grothendieck topologies.

Lemma 4.3.1. Consider a subclass $\mathcal{V} \subset \mathcal{U}$.

(i) The class of sequences

$$\{\sigma_u \otimes X : U \otimes U \otimes X \to U \otimes X \to X | u \in \mathcal{V}\},\$$

is a pretopology on \mathbf{a} , which we denote by $\mathcal{S}_{\mathcal{V}}$. We have $\widetilde{\mathcal{Co}}_1(\mathcal{S}_{\mathcal{V}}) = \overline{\mathcal{V}}$.

- (ii) The topology $\mathcal{T}_{\mathcal{V}} \coloneqq \operatorname{top}(\mathcal{S}_{\mathcal{V}})$ is noetherian. The topology is subcanonical if and only if $\mathcal{V} \subset \mathcal{U}^{ex}$ and monoidal if **a** is braided.
- (iii) The functor $Z : \mathbf{a} \to Sh(\mathbf{a}, \mathcal{T}_{\mathcal{V}})$ is faithful if and only if $\mathcal{V} \subset \mathcal{U}^{ep}$ and fully faithful if and only if $\mathcal{V} \subset \mathcal{U}^{ex}$. If $\mathcal{T}_{\mathcal{V}}$ is monoidal, the kernel of Z is $J_{\mathcal{V}}$.

Proof. For part (i), Condition (PTa) follows from the fact that for any $u \in \mathcal{V}$ and $f : A \to X$ in **a**, there is a commutative diagram

Condition (PTb) follows since for every $u: U \to \mathbf{1}$ in \mathcal{V} and $f: A \to U \otimes X$ with $(u \otimes X) \circ f = 0$, there is a commutative diagram

$$U \otimes U \otimes X \xrightarrow{u \otimes U \otimes X - U \otimes u \otimes X} U \otimes X$$

$$\downarrow U \otimes f \downarrow \qquad f \downarrow$$

For part (ii), that $\mathcal{T}_{\mathcal{V}}$ is noetherian follows from Lemma 3.1.8(i). That $\mathcal{T}_{\mathcal{V}}$ is subcanonical if and only if $\mathcal{V} \subset \mathcal{U}^{ex}$ follows from Theorem 3.2.2. That $\mathcal{T}_{\mathcal{V}}$ is monoidal when **a** is braided follows from Theorem 3.5.6.

The first sentence in part (iii) follows from Proposition 3.4.1 and Theorem 3.2.2. Now assume that $\mathcal{T}_{\mathcal{V}}$ is monoidal and let J denote the tensor ideal in **a** which is the kernel of Z. By Proposition 3.4.1, for each $W \in \mathbf{a}$ we have

$$J(\mathbf{1}, W) = \{ w : \mathbf{1} \to W | v \circ w = 0 \text{ for some } v \in \overline{\mathcal{V}} \}.$$

This implies $J(U, \mathbf{1}) = \operatorname{ann}_{\mathcal{V}}(U) = J_{\mathcal{V}}(U, \mathbf{1})$ by adjunction, so consequently $J = J_{\mathcal{V}}$.

Example 4.3.2. In [Co2], an object $X \in \mathbf{a}$ is called 'strongly faithful' if $\operatorname{ev}_X : X^* \otimes X \to \mathbf{1}$ satisfies (Ex). Note that ev_X satisfies (Ep) if and only if $X \otimes -$ is faithful. For \mathcal{V} the class of ev_X in \mathcal{U}^{ex} , the topology $\mathcal{T}_{\mathcal{V}}$ is studied in [Co2, §3].

Proposition 4.3.3. Consider a noetherian monoidal topology \mathcal{T} on \mathbf{a} such that for every epimorphism $N \twoheadrightarrow \mathbf{1}$ in $Sh(\mathbf{a}, \mathcal{T})$ from a rigid object N, the sequence $N \otimes N \to N \to \mathbf{1} \to 0$ is exact. Then $\mathcal{T} = \mathcal{T}_{\mathcal{V}}$, for \mathcal{V} the class of morphisms $u : U \to \mathbf{1}$ in \mathbf{a} for which Z(u) is an epimorphism in $Sh(\mathbf{a}, \mathcal{T})$.

Proof. By assumption and since the tensor product in $\mathsf{Sh}(\mathbf{a}, \mathcal{T})$ is right exact, the sequence $\mathsf{Z}(\sigma_u \otimes X)$ is exact for every $u \in \mathcal{V}$ and $X \in \mathbf{a}$. That $\mathcal{T} = \mathcal{T}_{\mathcal{V}}$ is an instance of Corollary 2.2.8(ii), by application of Lemma 4.2.5 to $h = \mathsf{Z}$.

Lemma 4.3.4. For $\mathcal{V} \subset \mathcal{U}$, the topology $\mathcal{T}_{\mathcal{V}}$ is monoidal if and only if for every $A \in \mathbf{a}$ and $u: U \to \mathbf{1}$ in \mathcal{V} , there exists $v: V \to \mathbf{1}$ in $\overline{\mathcal{V}}$ and a morphism $f: V \otimes A \to A \otimes U$ such that the following diagram is commutative



Proof. We apply Theorem 3.5.4(c). For fixed A, u, we can reformulate the existence of v, f as in the lemma as demanding that $A \otimes R_u$ is in $\mathcal{T}_{\mathcal{V}}(A)$. Hence the existence of such v, f are necessary for $\mathcal{T}_{\mathcal{V}}$ to be monoidal.

Conversely, assume that we always have v, f as in the lemma. For arbitrary $R \in \mathcal{T}_{\mathcal{V}}(X)$, by definition there exists $u \in \mathcal{V}$ with $u \otimes X \in R$. It follows that $v \otimes A \otimes X$ is in $A \otimes R$ and hence $A \otimes R \in \mathcal{T}_{\mathcal{V}}(A \otimes X)$. This concludes the proof.

Remark 4.3.5. (i) If for every $u: U \to \mathbf{1}$ in \mathcal{V} , there exists an object $(U, \gamma: -\otimes U \Rightarrow U \otimes -)$ of the Drinfeld centre of \mathbf{a} , then $\mathcal{T}_{\mathcal{V}}$ is monoidal.

(ii) For fixed $A \in \mathbf{a}$ and $u: U \to \mathbf{1}$ in \mathcal{V} , the condition in Lemma 4.3.4 is equivalent with the condition that we can complete the following commutative square



in a way that the left vertical arrow is in \mathcal{V} .

Proposition 4.3.6. Consider a subclass $\mathcal{V} \subset \mathcal{U}$ and assume that $\mathcal{T}_{\mathcal{V}}$ is monoidal, so that $\mathsf{Sh}(\mathbf{a},\mathcal{T}_{\mathcal{V}})$ is a biclosed Grothendieck category and $\mathsf{Z}: \mathbf{a} \to \mathsf{Sh}(\mathbf{a},\mathcal{T}_{\mathcal{V}})$ canonically monoidal.

(i) For another biclosed Grothendieck category C, composition with Z yields an equivalence

$$[\mathsf{Sh}(\mathbf{a},\mathcal{T}_{\mathcal{V}}),\mathbf{C}]_{cc}^{\otimes} \xrightarrow{\sim} [\mathbf{a},\mathbf{C}]_{\mathcal{V}}^{\otimes},$$

where $[\mathbf{a}, \mathbf{C}]_{\mathcal{V}}^{\otimes}$ stands for the category of monoidal functors F for which $F(\sigma_u)$ is exact in \mathbf{C} , for every $u \in \mathcal{V}$.

- (ii) The sequence $Z(\sigma_u)$ is exact in $Sh(\mathbf{a}, \mathcal{T}_{\mathcal{V}})$ for every $u \in \mathcal{V}$.
- (iii) If X is a scheme over k, then $[Sh(\mathbf{a}, \mathcal{T}_{\mathcal{V}}), QCohX]_{cc}^{\otimes}$ is equivalent to the category of monoidal functors $F : \mathbf{a} \to QCohX$ for which F(u) is an epimorphism, for every $u \in \mathcal{V}$.

Proof. Part (i) is a special case of Theorem 3.5.5. Part (ii) follows from Corollary 2.3.5. Part (iii) follows from part (i) and Remark 4.2.3. \Box

Lemma 4.3.7. For $\mathcal{V} \subset \mathcal{U}$, the following are equivalent:

(a) $\mathcal{T}_{\mathcal{V}} = \mathcal{T}_{\mathcal{U}}$.

(b) For every non-zero $u: U \to \mathbf{1}$, there exists $v: V \to \mathbf{1}$ in $\overline{\mathcal{V}}$ and $f: V \to U$ with $v = u \circ f$. In that case we say that \mathcal{V} is dense.

Proof. Since $\mathcal{T}_{\mathcal{V}} \subset \mathcal{T}_{\mathcal{U}}$, this is an almost immediate application of Corollary 2.2.8(i).

Remark 4.3.8. In [Co2, Analogy 3.2.3], a simplistic non-enriched analogue is given of $\mathsf{Sh}(\mathbf{D}, \mathcal{T}_{\mathcal{V}})$ for $\mathcal{V} \subset \mathcal{U}^{ex}$.

4.4. Universal properties.

Theorem 4.4.1. Let \mathcal{T} be a monoidal Grothendieck topology on \mathbf{a} for which $\mathsf{Sh}(\mathbf{a}, \mathcal{T}) \simeq \mathsf{Ind}\mathbf{T}$ for a tensor category \mathbf{T} . Then we have $\mathcal{T} = \mathcal{T}_{\mathcal{V}}$, with $\mathcal{V} = \{u \in \mathcal{U} | Z(u) \neq 0\}$, and $J_{\mathcal{V}}$ is the kernel of Z, so in particular $\operatorname{ann}_{\mathcal{V}}(U) = \{u : U \rightarrow \mathbf{1} | Z(u) = 0\}$. Furthermore, for any other tensor category \mathbf{T}_1 , composing with $\mathbf{a} \rightarrow \mathbf{T}$ induces an equivalence

$$[\mathbf{T},\mathbf{T}_1]_{ex}^{\otimes} \xrightarrow{\sim} [\mathbf{a}/J_{\mathcal{V}},\mathbf{T}_1]_{faith}^{\otimes}$$

between the respective categories of monoidal functors which are exact or faithful.

Proof. By Theorem 4.2.2(ii), that $\mathcal{T} = \mathcal{T}_{\mathcal{V}}$ is an instance of Proposition 4.3.3. That $J_{\mathcal{V}}$ is the kernel of Z follows from Lemma 4.3.1(iii).

Composition with the relevant monoidal functors yield equivalences

$$[\mathbf{T}, \mathbf{T}_1]_{ex}^{\otimes} \simeq [\mathsf{Ind}\mathbf{T}, \mathsf{Ind}\mathbf{T}_1]_{cc}^{\otimes} \simeq [\mathbf{a}, \mathsf{Ind}\mathbf{T}_1]_{\mathcal{V}}^{\otimes}$$

Indeed, the right equivalence is Proposition 4.3.6(i). Furthermore, Theorem 3.5.5 establishes an equivalence between the category in the middle and the category of right exact monoidal functors $\mathbf{T} \rightarrow \mathsf{Ind}\mathbf{T}_1$. By [De, Corollaire 2.10(i)], right exact monoidal functors $\mathbf{T} \rightarrow \mathsf{Ind}\mathbf{T}_1$ are automatically exact. Since the category of rigid objects in $\mathsf{Ind}\mathbf{T}_1$ is equivalent to \mathbf{T}_1 , see [Co2, Lemma 1.3.7], the equivalence follows.

The latter observation also shows that $[\mathbf{a}, \mathsf{Ind}\mathbf{T}_1]^{\otimes}_{\mathcal{V}}$ is equivalent to the category of functors in $[\mathbf{a}, \mathbf{T}_1]^{\otimes}$ which send every sequence $\sigma_u, u \in \mathcal{V}$, to an exact sequence. By Theorem 4.2.2(ii) the latter equals the subcategory of $F \in [\mathbf{a}, \mathbf{T}_1]^{\otimes}$ such that $F(v) \neq 0$ (or equivalently F(v) is surjective) for each $v \in \mathcal{V} = \overline{\mathcal{V}}$. By definition of $\operatorname{ann}_{\mathcal{V}}$, the latter condition on F also implies that F(w) = 0 for each $w \in \operatorname{ann}_{\mathcal{V}}$. By the above, we know that every morphism to $\mathbf{1}$ in \mathbf{a} is either in \mathcal{V} or $\operatorname{ann}_{\mathcal{V}}$. The condition on F is thus that its kernel is precisely $J_{\mathcal{V}}$. This concludes the proof.

Corollary 4.4.2. The following are equivalent for $\mathcal{V} \subset \mathcal{U}^{ex}$ with $\mathcal{T}_{\mathcal{V}}$ monoidal.

- (i) **a** is a full subcategory of a tensor category such that every object in the tensor category is a quotient of an object in **a**, and \mathcal{V} is dense.
- (ii) $Sh(\mathbf{a}, \mathcal{T}_{\mathcal{V}})$ is the ind-completion of a tensor category.

Furthermore, either condition implies $\mathcal{U} = \mathcal{U}^{ex}$.

Remark 4.4.3. By Theorem 4.4.1, if for a topology \mathcal{T} we have $\mathsf{Sh}(\mathbf{a}, \mathcal{T}) \simeq \mathsf{Ind}\mathbf{T}$ such that $\mathbf{a} \to \mathbf{T}$ is faithful, then $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$. Furthermore, a necessary condition is $\mathcal{U} = \mathcal{U}^{ep}$ by Lemma 4.3.1(iii). Finally, by Lemma 3.4.4(ii), \mathbf{T} will be a tensor category over k when $\mathbf{a}(\sigma_u, \mathbf{1})$ is exact for every non-zero $u : U \to \mathbf{1}$ in \mathbf{a} , *i.e.* when

$$0 \to \mathbf{a}(1,1) \to \mathbf{a}(U,1) \to \mathbf{a}(U \otimes U,1)$$

is always exact.

Corollary 4.4.4. Consider a monoidal functor $u : \mathbf{a} \to \mathbf{T}$ to a tensor category \mathbf{T} and denote its kernel by J. Assume that u satisfies (F) and (G) in Lemma 4.1.3. Then composition with $\mathbf{a}/J \to \mathbf{T}$ induced from u yields, for every tensor category \mathbf{T}_1 , an equivalence

$$[\mathbf{T},\mathbf{T}_1]_{ex}^{\otimes} \xrightarrow{\sim} [\mathbf{a}/J,\mathbf{T}_1]_{faith}^{\otimes}$$

Proof. By Lemma 4.1.3, $\mathbf{a}/J \rightarrow \mathbf{T}$ is a creator. The result is thus an immediate application of Theorem 4.4.1.

Corollary 4.4.5. Consider monoidal creators $u_i : \mathbf{a} \to \mathbf{T}_i$, for tensor categories \mathbf{T}_i , with $i \in \{1, 2\}$. The kernels of u_1 and u_2 are either equal or incomparable for the inclusion order.

Proof. By adjunction, the kernels of u_1, u_2 are determined by the kernels of $\mathbf{a}(-, \mathbf{1}) \rightarrow \mathbf{T}_i(-, \mathbf{1})$. Denote by \mathcal{V}_i the class of morphisms $U \rightarrow \mathbf{1}$ which are *not* in the kernel of $\mathbf{a}(-, \mathbf{1}) \rightarrow \mathbf{T}_i(-, \mathbf{1})$. Assume the kernel of u_1 is contained in the kernel of u_2 . This means that $\mathcal{V}_1 \supset \mathcal{V}_2$, so also $\operatorname{ann}_{\mathcal{V}_1} \supset \operatorname{ann}_{\mathcal{V}_2}$. By Theorem 4.4.1 the kernels are given (up to the above adjuntion) by $\operatorname{ann}_{\mathcal{V}_1}$ and $\operatorname{ann}_{\mathcal{V}_2}$, so by assumption we also have $\operatorname{ann}_{\mathcal{V}_1} \subset \operatorname{ann}_{\mathcal{V}_2}$, whence the kernels of u_1 and u_2 are equal.

4.5. Graded modules and projective space.

4.5.1. Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be a commutative and \mathbb{Z} -graded k-algebra, such that $k \to S_0$ is an isomorphism, S_1 is finite dimensional and S is generated in degree 1. We exclude the trivial case k = S.

We consider the category S-gMod of all Z-graded S-modules with homogeneous morphisms of degree zero, which is symmetric monoidal for \otimes_S . Then the full subcategory of finitely generated free modules constitutes a symmetric rigid monoidal category **a** over k with finite dimensional morphism spaces. The indecomposable objects in **a** are the modules $S\langle j \rangle$, for $j \in \mathbb{Z}$, which satisfy $S\langle j \rangle_i = S_{i-j}$. We denote by $\mathcal{V} \subset \mathcal{U}(\mathbf{a})$ the morphisms to the unit S for which the cokernel in S-gMod is finite dimensional.

Proposition 4.5.2. (i) There are no faithful monoidal functors from **a** to any tensor category.

(ii) Let \mathbb{Y} be the projective k-scheme $\operatorname{Proj}S$, then $\operatorname{QCoh}\mathbb{Y} \simeq \operatorname{Sh}(\mathbf{a}, \mathcal{T}_{\mathcal{V}})$.

Proof. Clearly, every indecomposable object in **a** is invertible. However there exist non-zero morphisms between non-isomorphic indecomposable objects. As invertible objects in tensor categories are necessarily simple, part (i) follows.

It is known, see [Se, §3], that QCoh \mathbb{Y} is the localisation of S-gMod \simeq PSha with respect to the subcategory of torsion modules. We will use that a topology on an additive category is determined by the sieves on indecomposable objects (it suffices to apply Theorem 1.4.5 to **a** and its category \mathbf{a}_0 of indecomposable objects). By Theorem 1.4.5, the topology \mathcal{T} corresponding to the localisation is given by all graded submodules $R \subset S\langle i \rangle$ for which $S\langle i \rangle/R$ is a torsion module, *i.e.* finite dimensional. Whence $\mathcal{T} = \mathcal{T}_{\mathcal{V}}$, proving part (ii). **Proposition 4.5.3.** Consider $S = k[x_0, x_1, \dots, x_n]$ for some $n \in \mathbb{N}$, with grading defined by $\deg x_i = 1$.

(i) We have $\mathcal{U}^{ex} \subset \mathcal{U}^{ep} = \mathcal{U}$, where the inclusion is strict.

(ii) If n = 1, then $\mathcal{U}^{ex} = \mathcal{V}$.

Proof. It is clear that every non-zero morphism $S \to ?$ in **a** is a monomorphism (it is even a monomorphism in S-gMod). It follows that $\mathcal{U}^{ep} = \mathcal{U}$. Next, we claim that a non-zero morphism $u: U \to S$ is in \mathcal{U}^{ex} if and only if imu is not included in a non-trivial principal homogeneous ideal. Both parts then follow easily.

An arbitrary $u: U \to S$ can be written as follows. Consider a finite collection of natural numbers $\{n_i\}$ and define

$$u: U \coloneqq \bigoplus_i S\langle n_i \rangle \to S, \quad (p_i)_i \mapsto \sum_i a_i p_i,$$

for some $a_i \in S_{n_i}$. Then im u is not contained in any non-trivial principal ideal if and only if the greatest common denominator of $\{a_i\}$ is 1.

Fix a morphism

$$f: U \to S\langle j \rangle, \quad (p_i)_i \mapsto \sum_i b_i p_i$$

with $j \in \mathbb{Z}$, defined by $b_i \in S_{n_i-j}$. Then $f \circ (U \otimes u - u \otimes U) = 0$ if and only if $b_i a_l = b_l a_i$ for all i, l. The latter is equivalent to the condition that there exists a rational function q such that $q = b_i/a_i$ for all i. On the other hand, we have that f factors as $U \xrightarrow{u} S \to S\langle j \rangle$ if and only if b_i/a_i is a polynomial, independent of i.

Now assume that the a_i have greatest common denominator $a \neq 1$ and set $b_i \coloneqq a_i/a$, so $b_i/a_i = 1/a$. By the above paragraph, this yields a morphism $f : U \to S \langle \deg a \rangle$ which contradicts exactness of σ_u . On the other hand, if the greatest common denominator of the a_i is 1, the above paragraph shows similarly that the sequence is exact.

Lemma 4.5.4. Let S be as in Proposition 4.5.3 and let $v : S(1)^{\oplus n+1} \to S$ be the morphism corresponding to $\{S(1) \to S : 1 \mapsto x_i \mid 0 \le i \le n\}$. Then $\mathcal{T}_{\mathcal{V}} = \mathcal{T}_{\{v\}}$.

Proof. This application of Corollary 2.2.8 is left as an exercise.

The following is a categorical analogue of the usual description of morphisms $\mathbb{X} \to \mathbb{P}^n$.

Proposition 4.5.5. Assume n > 0. For any scheme X we have an equivalence between

$$[\mathsf{QCoh}\mathbb{P}^n,\mathsf{QCoh}\mathbb{X}]^{\otimes,b}_{cc}$$

and the following category. An object ϕ is a collection of morphisms $\{\phi_i : \mathcal{L} \to \mathbf{1} = \mathcal{O}_{\mathbb{X}} \mid 0 \leq i \leq n\}$ from an invertible sheaf $\mathcal{L} \in \mathbb{Q}$ Coh \mathbb{X} such that $\bigoplus_{i=0}^{n} \mathcal{L} \to \mathbf{1}$ is an epimorphism. A morphism from $\{\phi_i : \mathcal{L} \to \mathbf{1}\}$ to $\{\psi_i : \mathcal{L}' \to \mathbf{1}\}$ is an isomorphism $\mathcal{L} \to \mathcal{L}'$ leading to n + 1 commutative diagrams with $\{\phi_i, \psi_i \mid 0 \leq i \leq n\}$.

Proof. Let S be as in Proposition 4.5.3, in particular $\mathbb{P}^n = \operatorname{Proj} S$. Note that **a** is the universal (free) symmetric monoidal category over k generated by an invertible object S(1) and the n + 1 morphisms $S(1) \to \mathbf{1}$ from Lemma 4.5.4. By Proposition 4.5.2 and Lemma 4.5.4, we have $\operatorname{QCoh} \mathbb{P}^n \simeq \operatorname{Sh}(\mathbf{a}, \mathcal{T}_{\{v\}})$. The conclusion now follows from Proposition 4.3.6(iii).

Acknowledgement. The author thanks Pavel Etingof, Victor Ostrik and Bregje Pauwels for many interesting discussions. Several ideas in Section 2 owe their existence to [Sc1, Sc2]. The research was supported by ARC grant DP180102563.

ADDITIVE PRETOPOLOGIES

References

- [AR] J. Adámek, J. Rosický: Reflections in locally presentable categories. Arch. Math. (Brno) 25 (1989), no. 1-2, 89–94.
- [AGV] M. Artin, A. Grothendieck, and J.-L. Verdier, Théorie des topos et cohomologie étale des schémas, 1, Springer-Verlag, Berlin, 1972–73, (SGA4). Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270.
- [BQ] F. Borceux, C. Quinteiro: A theory of enriched sheaves. Cahiers Topologie Géom. Différentielle Catég. 37 (1996), no. 2, 145–162.
- [Co1] K. Coulembier: Tensor ideals, Deligne categories and invariant theory. Selecta Math. (N.S.) 24 (2018), no. 5, 4659–4710.
- [Co2] K. Coulembier: Monoidal abelian envelopes. To appear in Compositio Mathematica. arXiv:2003.10105.
- [Da] B. Day: A reflection theorem for closed categories. J. Pure Appl. Algebra 2 (1972), no. 1, 1–11.
- [De] P. Deligne: Catégories tannakiennes. The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.
- [EGNO] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik: Tensor categories. Mathematical Surveys and Monographs, 205. American Mathematical Society, Providence, RI, 2015.
- [EHS] I. Entova-Aizenbud, V. Hinich, V. Serganova: Deligne categories and the limit of categories Rep(GL(m|n)). Int. Math. Res. Not. IMRN 2020, no. 15, 4602–4666.
- [Fr] P. Freyd: Abelian categories. An introduction to the theory of functors. Harper's Series in Modern Mathematics Harper & Row, Publishers, New York 1964.
- [IK] G.B. Im, G.M. Kelly: A universal property of the convolution monoidal structure. J. Pure Appl. Algebra 43 (1986), no. 1, 75–88.
- [Ke] G.M. Kelly: Basic Concepts of Enriched Category Theory. Cambridge University Press, Lecture Notes in Mathematics 64, 1982.
- [Lo] W. Lowen: A generalization of the Gabriel-Popescu theorem. J. Pure Appl. Algebra 190 (2004), no. 1-3, 197–211.
- [LRS] W. Lowen, J. Ramos González, B. Shoikhet: On the tensor product of linear sites and Grothendieck categories. Int. Math. Res. Not. IMRN 2018, no. 21, 6698–6736.
- [Ma] S. MacLane: Categories for the working mathematician. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971.
- [PG] N. Popesco, P. Gabriel: Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes. C. R. Acad. Sci. Paris 258 (1964), 4188–4190.
- [Sc1] D. Schäppi: Ind-abelian categories and quasi-coherent sheaves. Math. Proc. Cambridge Philos. Soc. 157 (2014), no. 3, 391–423.
- [Sc2] D. Schäppi: Constructing colimits by gluing vector bundles. Adv. Math. 375 (2020), 107394, 85 pp.
- [Se] J.P. Serre: Faisceaux algébriques cohérents. Ann. of Math. (ii) 61 (1955), 197–278.
- [Ta] G. Tamme: Introduction to étale cohomology. Universitext. Springer-Verlag, Berlin, 1994.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, NSW 2006, AUSTRALIA *Email address:* kevin.coulembier@sydney.edu.au